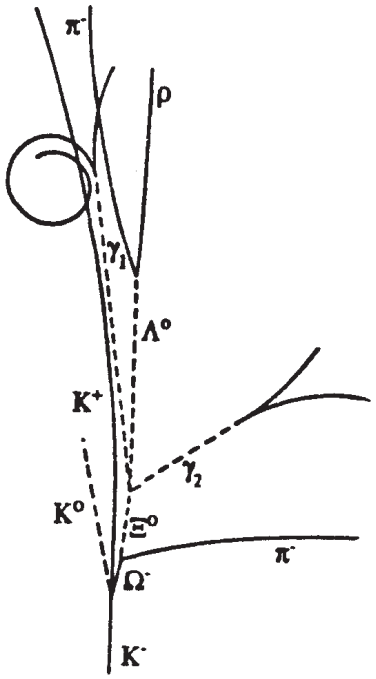
VOLUME 45, NUMBER 1, MARCH 2022

HADRONIC JOURNAL Volume 45, Number 1, March 2022

HADRONIC JOURNAL

Founded in 1978 by Prof. R. M. Santilli at Harvard University. Some of the past Editors include Professors S. L. Adler, A. 0. Barut, L. C. Biedenharn, N. N. Bogoliubov, M. Froissart, J.Lohmus, S. Okubo, Nobel Laureate Ilya Prigogine, M. L. Tomber, J. P.Vigier, J. Wess, Nobel Laureate Chen Ning Yang.



EDITORIAL BOARD

A.O.ANIMALU
A. K. ARINGAZIN
A. A. BHALEKAR
S. J. DHOBLE
J. DUNNING-DAVIES T.
L. GILL
L. P. HORWITZ
S. L. KALLA
S. I. KRUGLOV
M. NISHIOKA
R. F. O'CONNELL
Z. OZIEWICZ*
E. RECAMI
M. SALEEM
B. G. SIDHARTH
S. SILVESTROV
H. M. SRIVASTAVA
E. TRELL
R.I. TSONCHEV
QI-REN ZHANG
C.A.WOLF
YI-ZHONG ZHUO

** In memoriam*

FOUNDER and Editor In Chief
R. M. SANTILLI

HADRONIC PRESS, INC.

HADRONIC JOURNAL

Established in 1978 by Prof. R. M. Santilli at Harvard University Hadronic Journal and Algebras, Groups and Geometries have been regularly published since 1978 without publication charges and are among the few remaining independent refereed journals.

This Journal publishes advances research papers and Ph. D. theses in any field of mathematics without publication charges.

For subscription, format and any other information please visit the website
http://www.hadronicpress.com

HADRONIC PRESS INC.
35246 U. S. 19 North Suite 215
Palm Harbor, FL 34684, U.S.A.
http://www.hadronicpress.com
Email: info@hadronicpress.com
Phone: +1-727-946-0427

HADRONIC JOURNAL

Founder and Editor in Chief
RUGGERO MARIA SANTILLI
The Institute for Basic Research
35246 U. S. 19 North Suite 215, Palm Harbor, FL 34684, U.S.A.
Email: resarch@i-b-r.org; TEL: +1-727-688-3992

A. O. ANIMALU, University of Nigeria, Department of Physics, Nsukka, Nigeria animalu@nmc.edu.ng
A.K. ARINGAZIN Department of Theoretical Physics, Institute for Basic Research Eurasian National University Astana 010008, Kazakhstan aringazin@mail.kz
A.A. BHALEKAR Department of Chemistry, R.T.M. Nagpur University, Nagpur, 440033 India anabha@hotmail.com
S.J. DHOBLE Department of Physics R.T.M. Nagpur University Nagpur, 440033 India sjdhoble@rediffmail.com
J. DUNNING-DAVIES, Department of Physics (Retired) University of Hull Hull, HU6 7RX England j.dunning-davies@hull.ac.uk
T. L. GILL Howard University Research Center ComSERC Washington, DC 20059, USA tgill@howard.edu
L. P. HORWITZ, Department of Physics Tel Aviv Univ., Ramat Aviv, Israel horwitz@taunivm.tau.ac.il
S. L. KALLA, Department of Mathematics Vyas Institute of Higher Education Jodhpur, 342008, India shyamkalla@gmail.com
S.I. KRUGLOV University of Toronto at Scarborough, Physical and Environmental Sciences Dept., 1265 Military Trail, Toronto, Ontario, Canada M1C 1A4 skrouglo@utm.utoronto.ca
M. NISHIOKA, Yamaguchi University Department of Physics, Yamaguchi 753, Japan
R. F. O'CONNELL, Louisiana State University Department of Physics, Baton Rouge, LA 70803
Z.OZIEWICZ,* Universidad National Autonoma de Mexico, Faculdad de Estudios Superiores C.P. 54714, Cuautitlan Izcalli Aparto Postal # 25, Mexico **** In memoriam***
E. RECAMI, Universita' de Bergamo, Facolta' di Ingeneria, Viale Marconi 5, 1-24044 Dalmine (BG) Italy recami@mi.ifn.it
M. SALEEM, University of the Punjab Center for High Energy Phys., Lahore, Pakistan dms@lhr.paknet.com.pk
B.G. SIDHARTH International Institute for Applicable Mathematics &b Information Sciences, B.M. Birla Science Center, Adarshnagar, Hyderabad 500063 (India) iimisbgs@yahoo.com.in
S. SILVESTROV School of Education, Culture and Communication (UKK) Malardalen University Box 883, 71610 Västerås, Sweden sergei.silvestrov@mdh.se
H. M. SRIVASTAVA, Department of Mathematics and Statistics, University of Victoria, Victoria, B. C. V8W 3P4, Canada, hmsri@uvvm.uvic.ca
E. TRELL Faculty of Health Sciences, University of Linkoping, Se-581 83, Linkoping, Sweden erik.trell@gmail.com
R.I. TSONCHEV Facultad de Fisica, Universidad Autonoma de Zacatecas, P. O. C-580, Zacatecas 98068, zac, Mexico rumen@ahobon.reduaz.ms
QI-REN ZHANG, Peking University, Department of Technical Phys., Beijing 100871, China zhangqr@sun.ihep.ac.cn
C.A. WOLF Department of Physics, Massachusetts College of Liberal Arts, North Adams, Ma 01247 cwolf@mcla.mass.edu
YI-ZHONG ZHUO, Institute of Atomic Energy P.O. Box 275 (18) Beijing 102413, China zhuoyz@mipsa.ciae.ac.cn

ISSN: 0162-5519

Established in 1978 by Prof. R. M. Santilli at Harvard University Hadronic Journal and Algebras, Groups and Geometries have been regularly published since 1978 without publication charges and are among the few remaining independent refereed journals.

This Journal publishes advances research papers and Ph. D. theses in any field of mathematics without publication charges.

For subscription, format and any other information please visit the website http://www.hadronicpress.com

HADRONIC PRESS INC.
35246 U. S. 19 North Suite 215
Palm Harbor, FL 34684, U.S.A.
http://www.hadronicpress.com
Email: info@hadronicpress.com
Phone: +1-727-946-0427

HADRONIC JOURNAL Volume 45, Number 1, March 2022

H P HADRONIC PRESS, INC.

HADRONIC JOURNAL

VOLUME 45, NUMBER 1, MARCH 2022

A NEW DESCRIPTION OF THE COMPLEX NEUTRINO PROBLEM IN CONNECTION WITH THE COSMOLOGY, 1

H. Strusny

Hohenschoenhauser Str. 1

10369 Berlin, Germany

INVESTIGATE THE EFFECT OF TEMPERATURE ON QUASIPARTICLE EXPECTATION VALUE AND PAIR CORRELATION FUNCTION FOR SINGLET WAVE IN AN OPEN QUANTUM DISSPATIVE SYSTEM AT NON-EQUILIBRIUM, 59

Nabin Kumar Teemilsina[1,2], Saddam Husain Dhobi[1,2,3], Kishori Yadav[1,2], Bibek Koirala[1,2], Padam Prakash Ghimire[1,2], Dilip Rokka[1,2], Suresh Prasad Gupta[1,2], Jeevan Jyoti Nakarmi[1,2]

[1]Department of Physics, Patan Multiple Campus, Tribhuvan University, Lalitpur-44700, Nepal

[2]Innovative Ghar Nepal, Lalitpur-44700, Nepal

[3]Robotics Academy of Nepal, Lalitpur-44700, Nepal

KEPLER PROBLEM AND PERIHELION IN THE MODEL OF THE DYNAMIC QUANTUM VACUUM, 81

Davide Fiscaletti

SpaceLife Institute, Via Roncaglia, 35 – 61047 San Lorenzo in Campo (PU), Italy

Štefan Čelan

Scientific Research Centre Bistra, Slovenski trg 6, SI-2250 Ptuj, Slovenia

Amrit Šorli

Scientific Research Centre Bistra, Slovenski trg 6, SI-2250 Ptuj, Slovenia

NEW THERMODYNAMICS: THE SECOND LAW BURIED BY ILLUSIONS, 97

Kent Mayhew

68 Pineglen Cres.

Ottawa, Ontario, Canada K2G 0G8

HADRONIC JOURNAL 45, 1 - 58 (2022)
DOI 10.29083/HJ.45.01.2022

A NEW DESCRIPTION OF THE COMPLEX NEUTRINO PROBLEM IN CONNECTION WITH THE COSMOLOGY

H. Strusny
Hohenschoenhauser Str. 1
10369 Berlin, Germany
p.ressel@freenet.de

Received October 11, 2021

Abstract

We describe a new model of the complex neutrino problem in the particle physics and the cosmology. At the estimation of the 3 light neutrino rest energies, we compare a known old and a completely new approximation, using recent neutrino oscillation data. The new approximation is based on an extreme value solution for the sum of the 3 light neutrino rest energies. For the 3 heavy neutrinos, we estimate their rest energies and number densities via the 3 independent present cosmological particle density parameters of the universe. By aid of the properties of these heavy neutrinos, taking the reliably measured data of the photon decay of an unidentified X-ray energy line attributed to the sterile neutrinos as dark matter candidates, we determine the rest energies and the number densities of the 3 sterile neutrinos. We prove that these sterile neutrinos could be existent as one's own species. We explain exactly the presently observed Hubble constants, which are significantly greater than the Hubble constant of Planck 2013. We prove that the Hubble constant of Planck 2013 is precise and unique the Hubble constant of the present universe. The successful evaluation of the mean negative acceleration (Pioneer anomaly) and the astronomical unit changing as well as the flyby anomaly confirms our estimated values of the Hubble constants and the new model. These estimated Hubble constants show clearly that the "present" exponential expansion is decelerated and not accelerated. The end of this decelerated exponential expansion is connected with the final state and the big bang of the universe. The angular-diameter distance of the final state of the universe permits the unique conclusion that the new approximation is realistic in contrast to the old approximation.

Copyright © 2022 by Hadronic Press, Inc., Palm Harbor, FL 34684, USA

1 Introduction

Actually, the neutrino problem is dominating extremely for the particle physics in connection with the cosmology [1-36]. Therefore, in this work, because of the improved 3 light neutrino oscillation data [20], we investigate once more this neutrino problem, which was treated already by the works [1-6]. For this goal, we use a known old and a completely new approximation for the estimation of the 3 light neutrino rest energies. The old approximation [1-6] is based on a fictitious approach assumed for the first time in Ref. [1]. The new approximation, introduced in this work, uses an extreme value solution for the sum of the 3 light neutrino rest energies.

The sum of the 3 light neutrino rest energies of the old approximation lies near to the lower boundary of the expected energy range [20]. The new approximation gives a value for the sum of the 3 light neutrino rest energies, which does not lie in this expected energy range. However, it lies near to its lower boundary. In contrast to the old approximation, the new approximation gives a better agreement with the extrapolated SUSY Gut coupling constant $\alpha_{\mathrm{GUT}} \cong 0.04$ [18, 22-27], so that in this work α_{GUT} was estimated more exactly via the rest energies of the X, Y bosons and the magnetic monopoles, since the X, Y bosons are defined by aid of the 3 light neutrino rest energies.

In connection with the results of Refs. [2-6], we assume that the 3 independent present cosmological particle density parameters of the universe are coupled correspondingly with 3 heavy neutrinos, so that their rest energies and number densities can be determined. By aid of the properties of these heavy neutrinos, using the observed data of several researcher groups [28-33] for the photon decay of an unidentified X-ray emission line, assumed for the sterile neutrinos as dark matter candidates [28], we estimate the rest energies and the number densities of the 3 sterile neutrinos. We prove that these sterile neutrinos could be existent as one's own species in the universe.

Thus, we create a new description for the above-mentioned neutrino problem. In this connection, in the framework of the $\Lambda\mathrm{CDM}$, we explain more correctly the problem of the Hubble constants in contrast to work [6]. During the present "accelerated" exponential expansion, Hubble constants of

a mean value of about $73.24 \pm 1.74\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ were recently observed by Riess et al. [34]. They are greater than the value $H_0 = 67.4 \pm 0.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ of Planck 2013 [16], taken from Ref. [20]. According to Riess et al., this fact means a further expansion of the universe. However, we show that the Hubble constants decrease from a maximum value ($H_{\max} > H_0$) to a minimum value ($H_0 > H_{\min}$), whereat the value $H_0 = 67.4\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ lies in the middle of this range. The mean value $73.24 \pm 1.74\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ of the presently observed Hubble constants lies in the lower half of the upper range ($H_{\max} > H_0$). Consequently, this complete range of the estimated Hubble constants defines a "present" decelerated exponential expansion. We prove that the value $H_0 = 67.4 \pm 0.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ of Planck 2013 is precise and unique the present Hubble constant of the present universe.

By the successful evaluation of the mean negative acceleration (Pioneer anomaly [11, 12]) and the astronomical unit changing [13-15] as well as the flyby anomaly [35], our estimations of the Hubble constants are confirmed.

Their value $H_{\min}$ is connected with the final state of the universe. By the parameters of this final state, we show that we have a relationship between the big bang and final state of the universe, so that the most important results of our early works [1-7] are confirmed. Via the angular-diameter distance of this final state, we can prove that the new approximation of the estimation of the 3 light neutrino rest energies is realistic in contrast to their old approximation. In this connection, the assumption of a slow linear expansion [4-6] is incorrect because it describes only approximately the final state of the universe for the unrealistic old approximation.

Therefore. this work is organized as follows. In Sec. 2, we summarize the most recent data of the cosmological parameters. In Sec. 3, we estimate the 3 light neutrino rest energies via the old approximation. In Sec. 4, we evaluate the 3 light neutrino rest energies via the new approximation. In Sec. 5, by the results of the chapter 3 and 4, we calculate the SUSY GUT coupling constants by aid of the rest energies of the X, Y bosons and the magnetic monopoles. In Sec. 6, we mention the number density of the 3 light neutrinos. In Sec. 7, we

determine the corresponding 3 light neutrino density parameters of the universe for the two approximations. In Sec. 8, we evaluate the rest energies and the number densities of the 3 heavy neutrinos via the 3 independent present cosmological particle density parameters of the universe. In Sec. 9, we determine the properties of the sterile neutrinos. In Sec. 10, we solve exactly the problem of the Hubble constants. In Sec. 11, we estimate the mean negative acceleration (Pioneer anomaly) and the astronomical unit changing as well as the flyby anomaly. In Sec. 12, we evaluate the photon densities of the sterile neutrinos. In Sec. 13, we treat the relationship between the big bang and the final state of the universe. In Sec. 14, we determine directly the age of the final state of the universe via the sterile neutrinos. In Sec. 15, via the angular-diameter distance of the final state of the universe, we show that the new approximation of the estimation of the 3 light neutrino rest energies is realistic in contrast to their old approximation. In Sec. 16, we give a short summary. The values used for physical constants, are given in Refs. [17-20].

2 The most recent data of important cosmological parameters

The most recent observed data of important cosmological parameters, taken from Ref. [20}, are summarized in Table I.

TABLE I. The most recent observed data of important cosmological parameter values of the universe.

Quantity	**Symbol/equation**	**Value**
present day CMB temperature	T_0	$2.7255(6)\ \mathrm{K}$
age of the universe	t_0	$13.797(23)\ \mathrm{Gyr} =$ $= 4.3541(73)\cdot 10^{17}\ \mathrm{s}$

Table I. Continued,

Quantity	Symbol/equation	Value
scale factor of the Hubble expansion rate	h	0.674 ± 0.005
present day Hubble expansion rate	H_0 [a)]	$67.4 \pm 0.5\ \mathrm{km\ s^{-1}\,Mpc^{-1}} =$ $= h \times (9.777752\ \mathrm{Gyr})^{-1} =$ $= (2.184 \pm 0.016) \cdot 10^{-18}\ \mathrm{s^{-1}}$
critical present density of the universe	$\rho_{0C} =$ $= 3H_0^2 / 8\pi G_N$ [b)]	$1.053672(24) \cdot 10^{-5}\, h^2\, (\mathrm{GeV}/c^2)\ \mathrm{cm^{-3}} =$ $= 4.787(36) \cdot 10^3\, (\mathrm{eV}/c^2)\ \mathrm{cm^{-3}}$ [c)]
baryon (proton) density of the universe	$\Omega_b = \rho_b / \rho_{0C}$	$0.02237(15)\, h^{-2} = 0.0493(10)$
(cold) dark matter density of the universe	$\Omega_{dm} = \rho_{dm} / \rho_{0C}$	$0.1200(12)\, h^{-2} = 0.265(7)$
dark energy density of the universe	$\Omega_\Lambda = \rho_\Lambda / \rho_{0C}$	$0.685(7)$
pressureless matter density of the universe	$\Omega_m = \Omega_b + \Omega_{dm}$	$0.1424(14)\, h^{-2} = 0.315(7)$
CMB radiation density of the universe	$\Omega_\gamma = \rho_\gamma / \rho_{0C}$	$2.473 \cdot 10^{-5}\, (T/2.7255\ \mathrm{K})^4\, h^{-2} =$ $= 5.44(8) \cdot 10^{-5}$ [d)]

TABLE I. Continued.

Quantity	Symbol, equation	Value
Hubble length	$R_0 = c/H_0$	$0.9250629 \cdot 10^{26} h^{-1}$ m = $= 1.372(10) \cdot 10^{26}$ m
sum of neutrino masses	$\sum m_\nu c^2$	< 0.12 eV (95%, CMB + BAO), > 0.06 eV (mixing)
neutrino density of the universe	$\Omega_\nu = h^{-2} \sum m_\nu c^2 / 93.14\,\text{eV}$	< 0.003 (95%, CMB + BAO), > 0.0012 (mixing)
redshift of matter-radiation equality	z_{eq}	3402(26)

[a] H_0 is confirmed as the present-day Hubble expansion rate of the universe

[b] G_N denotes the gravitation constant.

[c] c describes the speed of light in vacuum.

[d] Calculated for $h = 0.674 \pm 0.005$.

3 The 3 light neutrino rest energies of the old approximation

There are three neutrinos (tauon neutrino ν_τ, muon neutrino ν_μ and electron neutrino ν_e).

According to Zyla et al. [20], the new 3-neutrino oscillation data yield for the square mass differences of the neutrino rest energies:

$$\Delta m_{\tau\mu}^2 c^4 = E_0^2(\nu_\tau) - E_0^2(\nu_\mu) = (2.453 \pm 0.034) \cdot 10^{-3}\ \text{eV}^2, \qquad (3.1)$$

$$\Delta m_{\mu e}^2 c^4 = E_0^2(\nu_\mu) - E_0^2(\nu_e) = (7.53 \pm 0.18) \cdot 10^{-5}\ \mathrm{eV}^2 . \tag{3.2}$$

From Eqs. (3.1) and (3.2), we can form generally the equations

$$E_0(\nu_\tau) = \left(\Delta m_{\tau\mu}^2 c^4 + E_0^2(\nu_\mu)\right)^{1/2} =$$

$$= \left(\Delta m_{\tau\mu}^2 c^4 + \Delta m_{\mu e}^2 c^4 + E_0^2(\nu_e)\right)^{1/2} \tag{3.3}$$

and

$$E_0(\nu_\mu) = \left(\Delta m_{\mu e}^2 c^4 + E_0^2(\nu_e)\right)^{1/2} . \tag{3.4}$$

Using at normal hierarchy the old approximation with the fictitious approach

$$E_0(\nu_\mu) \cong \left[E_0(\nu_\tau)\, E_0(\nu_e)\right]^{1/2}, \tag{3.5}$$

so that by Eqs. (3.3) and (3.4) via Eq. (3.5) we obtain

$$E_0(\nu_e) \cong \frac{E_0^2(\nu_\mu)}{E_0(\nu_\tau)} = \frac{\Delta m_{\mu e}^2 c^4}{\left[\Delta m_{\tau\mu}^2 c^4 - \Delta m_{\mu e}^2 c^4\right]^{1/2}} =$$

$$\cong (1.544 \pm 0.049) \cdot 10^{-3}\ \mathrm{eV} . \tag{3.6}$$

Consequently, by Eq. (3.6), the expressions (3.3) and (3.4) yield

$$E_0(\nu_\tau) = \left(\Delta m_{\tau\mu}^2 c^4 + \Delta m_{\mu e}^2 c^4 + E_0^2(\nu_e)\right)^{1/2} \cong$$

$$\cong (5.031 \pm 0.036) \cdot 10^{-2}\ \mathrm{eV} \tag{3.7}$$

and

$$E_0(\nu_\mu) = \left(\Delta m_{\mu e}^2 c^4 + E_0^2(\nu_e)\right)^{1/2} \cong (8.81 \pm 0.11) \cdot 10^{-3}\ \mathrm{eV} . \tag{3.8}$$

Then, for the old approximation, the sum of the neutrino rest energies [see Eqs. (3.6) to (3.8)] gives

$$\sum_i E_0(\nu_i) = E_0(\nu_\tau) + E_0(\nu_\mu) + E_0(\nu_e) \cong$$

$$\cong (6.066 \pm 0.052) \cdot 10^{-2}\ \mathrm{eV}, \tag{3.9}$$

where the subscript $i = \mathrm{e}, \mu, \tau$ characterizes the e, μ and τ neutrino.

This sum $\sum_i E_0(\nu_i) \cong (6.066 \pm 0.052) \cdot 10^{-2}\ \mathrm{eV}$ [see Eq. (3.9)] of the old approximation lies near to the lower boundary in the expected energy range $0.12\ \mathrm{eV} > \sum_i E_0(\nu_i) > 0.06\ \mathrm{eV}$ [20].

4 The 3 light neutrino rest energies of the new approximation

Generally, assuming $E_0(\nu_e) << \sqrt{\Delta m^2_{\mu e} c^4}$, instead of Eqs (3.7) and (3.8), we can write

$$E_0(\nu_\tau) = \left(\Delta m^2_{\tau\mu} c^4 + \Delta m^2_{\mu e} c^4 + E_0^2(\nu_e)\right)^{1/2} \cong$$

$$\cong \left(\Delta m^2_{\tau\mu} c^4 + \Delta m^2_{\mu e} c^4\right)^{1/2} \left(1 + \frac{1}{2} \frac{E_0^2(\nu_e)}{\Delta m^2_{\tau\mu} c^4 + \Delta m^2_{\mu e} c^4}\right) \tag{4.1}$$

and

$$E_0(\nu_\mu) = \left(\Delta m^2_{\mu e} c^4 + E_0^2(\nu_e)\right)^{1/2} \cong$$

$$\cong \left(\Delta m^2_{\mu e} c^4\right)^{1/2} \left(1 + \frac{1}{2} \frac{E_0^2(\nu_e)}{\Delta m^2_{\mu e} c^4}\right). \tag{4.2}$$

Then, using the negative sign of the root $\sqrt{\Delta m^2_{\mu e} c^4} = \pm \Delta m_{\mu e} c^2$ to obtain a positive value $E_0(\nu_e)$, the sum of the 3 light neutrino rest energies can be expressed by

$$\sum_i E_0(\nu_i) = \left[\left(\Delta m^2_{\tau\mu} c^4 + \Delta m^2_{\mu e} c^4\right)^{1/2} - \Delta m_{\mu e} c^2\right] +$$

$$+ \frac{1}{2}\left[\frac{1}{\left(\Delta m^2_{\tau\mu} c^4 + \Delta m^2_{\mu e} c^4\right)^{1/2}} - \frac{1}{\Delta m_{\mu e} c^2}\right] E_0^2(\nu_e) + E_0(\nu_e). \tag{4.3}$$

By the condition

$$\frac{d\sum_i E_0(\nu_i)}{dE_0(\nu_e)} = 0, \tag{4.4}$$

Eq. (4.3) gives

$$E_0(\nu_e) = \frac{\left(\Delta m^2_{\tau\mu} c^4 + \Delta m^2_{\mu e} c^4\right)^{1/2} \Delta m_{\mu e} c^2}{\left(\Delta m^2_{\tau\mu} c^4 + \Delta m^2_{\mu e} c^4\right)^{1/2} - \Delta m_{\mu e} c^2}. \tag{4.5}$$

Taking the values (3.1) and (3.2), the result (4.5) yields

$$E_0(\nu_e) \cong (5.74 \pm 0.19) \cdot 10^{-5}\,\mathrm{eV}, \tag{4.6}$$

so that the expressions (3.7) and (3.8) provide

$$E_0(\nu_\tau) = \left(\Delta m^2_{\tau\mu} c^4 + \Delta m^2_{\mu e} c^4 + E_0^2(\nu_e)\right)^{1/2} \cong$$

$$\cong (5.028 \pm 0.036) \cdot 10^{-2}\,\mathrm{eV} \tag{4.7}$$

and

$$E_0(\nu_\mu) = \left(\Delta m^2_{\mu e} c^4 + E_0^2(\nu_e)\right)^{1/2} \cong (8.68 \pm 0.10) \cdot 10^{-3}\,\mathrm{eV}. \tag{4.8}$$

Then, for the new approximation, the sum of the neutrino rest energies [see Eqs. (4.6) to (4.8)] gives

$$\sum_i E_0(\nu_i) = E_0(\nu_\tau) + E_0(\nu_\mu) + E_0(\nu_e) \cong$$

$$\cong (5.904 \pm 0.046) \cdot 10^{-2}\,\mathrm{eV}. \tag{4.9}$$

This sum (4.9) of the new approximation does not lie in the expected energy range $0.12\,\mathrm{eV} > \sum_i E_0(\nu_i) > 0.06\,\mathrm{eV}$ [20]. However, it lies near to its lower boundary. This fact is not critical, since this lower boundary is an assumed value as first approximation in Ref. [20].

Consequently, the expected energy range cannot solve the problem of the correct approximation. Therefore, in the next chapter, we estimate the SUSY GUT coupling constants of the two approximations.

5 The SUSY GUT coupling constants of the two approximations

According to Refs. [1-6, 22], the SUSY GUT coupling constant is defined via the ratio of the rest energies $E_0(\mathrm{X})$ of the X gauge bosons and $E_0(\mathrm{M})$ of the magnetic monopoles, i.e.

$$\alpha_{\mathrm{GUT}} = \frac{E_0(\mathrm{X})}{E_0(\mathrm{M})}. \tag{5.1}$$

Therefore, we determine first the rest energy of the X gauge boson. For the Standard $\mathrm{SU}(2) \times \mathrm{U}(1)$ electroweak theory, Fujikawa and Shrock [21] (see also Ref. [17]) have derived the magnetic moment of massive neutrinos to

$$\mu_\nu(\nu_i) = \frac{3}{8\pi^2\sqrt{2}} e[G_F/(\hbar c)^3]\hbar c^2\, E_0(\nu_i) =$$

$$= 3.203 \cdot 10^{-19} (E_0(\nu_i)/\mathrm{eV})\, \mu_B\,, \qquad (5.2)$$

where e, $[G_F/(\hbar c)^3] = 1.1663787 \cdot 10^{-23}\,\mathrm{eV}^{-2}$ and $\hbar$ describe the elementary charge, the Fermi coupling constant and the reduced Planck constant. The Bohr magneton μ_B [17] is defined by

$$\mu_B = \frac{e\hbar c^2}{2E_0(\mathrm{e})} = (5.7883818066(38) \cdot 10^{-11}\,\mathrm{MeV/T}\,, \qquad (5.3)$$

where $E_0(\mathrm{e})$ is the rest energy of the electron (e) (see, e.g., Ref. [17]) The magnetic moment of the electron is given by $\mu_\mathrm{e} = \mathrm{C_e}\mu_B$, where $\mathrm{C_e} \cong 1.00116$ [17]. Similar to the nucleon magneton $\mu_N = (E_0(\mathrm{e})/E_0(\mathrm{p}))\mu_B$ [17], where $E_0(\mathrm{p})$ describes the proton rest energy [17], the X magneton (SUSY grand unification magneton) is assumed to

$$\mu_\mathrm{X} = \frac{E_0(\mathrm{e})}{E_0(\mathrm{X})}\mu_B\,, \qquad (5.4)$$

so that analogous to the magnetic moment of the electron (see above) the magnetic moment of the neutrinos, using reasonably a neutrino constant $\mathrm{C}_\nu \geq \mathrm{C_e}$ in contrast to Refs. [1-6], may be assumed to

$$\mu_\nu(\nu_i) \approx \mathrm{C}_\nu \frac{E_0(\nu_i)}{\sum_i E_0(\nu_i)}\mu_\mathrm{X} = \mathrm{C}_\nu \frac{E_0(\nu_i)}{\sum_i E_0(\nu_i)} \frac{E_0(\mathrm{e})}{E_0(\mathrm{X})}\mu_B\,. \qquad (5.5)$$

Then, the derivation of Eq. (5.5) is clear because the expression $\sum_i \mu_\nu(\nu_i)/\mathrm{C}_\nu$ gives again the grand unification magneton μ_X [see Eq. (5.4)]. By Eqs. (5.2) and (5.5), if we assume the value $\mathrm{C}_\nu \cong \mathrm{C_e}$ in Eq. (5.5), we may estimate the rest energy of the X gauge boson to

$$E_0(\mathrm{X}) \cong \mathrm{C_e} \frac{E_0(\nu_i)}{\sum_i E_0(\nu_i)} \frac{E_0(\mathrm{e})}{\mu_\nu(\nu_i)}\mu_B =$$

$$= \mathrm{C_e} \frac{E_0(\mathrm{e})}{\sum_i E_0(\nu_i)} \frac{\mathrm{eV}}{3.203 \cdot 10^{-19}}\,. \qquad (5.6)$$

Then, using the results (3.9) and (4.9), we obtain for the old approximation

$$E_0(\mathrm{X}) = (2.633 \pm 0.023) \cdot 10^{16}\ \mathrm{GeV} \tag{5.7}$$

and for the new approximation

$$E_0(\mathrm{X}) = (2.705 \pm 0.021) \cdot 10^{16}\ \mathrm{GeV}, \tag{5.8}$$

respectively.

The values (5.7) and (5.8) agree excellently with the gauge (g) coupling unification at $E_{\mathrm{g}} \approx 3 \times 10^{16}\,\mathrm{GeV}$, which is identical with $E_0(\mathrm{X}) = E_{\mathrm{g}}$ and $E_0(\mathrm{Y}) = E_{\mathrm{g}}$ (see, e.g., Ref. [22]), so that we can assume $E_0(\mathrm{X}) = E_0(\mathrm{Y})$.

Now, we estimate the rest energy of the magnetic monopoles. For this goal, we use the new inflation model [1-6], which is defined as follows. By the present critical density $\rho_{0\mathrm{C}}$ (see Table I), we can introduce

$$\rho_{0\mathrm{C}} = \frac{3H_0^2}{8\pi G_N}\left[\frac{R_{\mathrm{Pl}}^2}{R_{\mathrm{Pl}}^2}\right] = \frac{3H_0^2}{8\pi G_N}\left[\frac{\hbar G_N/c^3}{R_{\mathrm{Pl}}^2}\right] = \\ = \frac{3\hbar}{8\pi c\,(R_{\mathrm{Pl}}\,c/H_0)^2} = \frac{3\hbar}{8\pi c\,\widetilde{R}_0^4}, \tag{5.9}$$

where the Planck radius is defined by $R_{\mathrm{Pl}} = \sqrt{\hbar G_N/c^3}$, so that the scale factor $\widetilde{R}_0$ is given by

$$\widetilde{R}_0 = (R_{\mathrm{Pl}}\,c/H_0)^{1/2} = (4.710 \pm 0.018) \cdot 10^{-5}\ \mathrm{m}. \tag{5.10}$$

Similar to Eq. (5.9), we can also introduce

$$\rho_{0\mathrm{C}} = \frac{3H_0^2}{8\pi G_N}\left[\frac{c^2}{c^2}\right] = \frac{3}{8\pi G_N}\frac{c^2}{[c/H_0]^2} = \frac{3c^2}{8\pi G_N R_0^2} \tag{5.11}$$

with

$$R_0 = c/H_0 = (1.372 \pm 0.010) \cdot 10^{26}\ \mathrm{m}. \tag{5.12}$$

Between Eqs. (5.10) and (5.12), we have trivially the connection $\widetilde{R}_0 = (R_{\mathrm{Pl}}\,R_0)^{1/2}$. Because of $\widetilde{R}_0 << R_0$, we assume that Eqs. (5.10) and (5.12) represent the "two" unknown scale factors of the absorption of light at the redshift condition [1-6] before and after the inflation, respectively.

Using the results (5.10) and (5.12), we can define

$$1+z_{\mathrm{M}}=\frac{\widetilde{R}_0}{R_{\mathrm{Pl}}}=\frac{R_0}{\widetilde{R}_0}=\frac{T_{\mathrm{M}}}{T_0}=(2.914\pm 0.011)\cdot 10^{30}. \tag{5.13}$$

In Eq. (5.13), because of $E_0(\mathrm{M})>E_0(\mathrm{X})$, the subscript M is assumed to define the magnetic monopoles, so that we can determine their temperature as follows:

$$T_{\mathrm{M}}=(1+z_{\mathrm{M}})T_0=(7.942\pm 0.032)\cdot 10^{30}\ \mathrm{K}. \tag{5.14}$$

Consequently, the rest energy of the magnetic monopoles is defined by

$$E_0(\mathrm{M})=kT_{\mathrm{M}}=(1+z_{\mathrm{M}})kT_0=(6.844\pm 0.028)\cdot 10^{17}\ \mathrm{GeV}. \tag{5.15}$$

Thus, because of the results (5.10) as well as (5.12) and (5.13), the scale factors $\widetilde{R}_0$ and R_0 are two characteristic quantities for the redshift or blueshift condition, so that the two scale factors $\widetilde{R}$ and R for the emission of light can be assumed for every z value as follows:

$$\widetilde{R}=\frac{R}{1+z_{\mathrm{M}}}=\frac{\widetilde{R}_0}{1+z}\quad(\text{see Eq. (5.17)}) \tag{5.16}$$

and

$$R=(1+z_{\mathrm{M}})\widetilde{R}=\frac{R_0}{1+z}\quad(\text{see Eq. (5.16)}). \tag{5.17}$$

Consequently, in connection with Eqs. (5.13) and (5.16), the result (5.17) shows clearly, that the scale factors of the early universe ($R_{\mathrm{Pl}}\le\widetilde{R}\le\widetilde{R}_0$) enlarge by the enormous factor $1+z_{\mathrm{M}}=(2.914\pm 0.011)\cdot 10^{30}$ to the scale factors of the late universe ($\widetilde{R}_0\le R\le R_0$). This fact represents our new inflation model [1-6].

Then, by Eq. (5.1), using the results (5.7) and (5.8) as well as (5.15), the SUSY GUT coupling constants are for the old approximation

$$\alpha_{\mathrm{GUT}}=0.03847\pm 0.00049 \tag{5.18}$$

and for the new approximation

$$\alpha_{\mathrm{GUT}}=0.03952\pm 0.00047. \tag{5.19}$$

Consequently, in contrast to the old approximation, the new approximation gives a better agreement with the SUSY Gut coupling constant of

$\alpha_{GUT} \cong 0.04$, obtained by the extrapolation to the unification of the 3 low energy gauge couplings of the strong and electroweak interaction at very high energies of about $3 \cdot 10^{16}$ GeV [18, 22-27]. Therefore, the new approximation must be more realistic. To this problem, we will return in chapters 14 and 15.

6 The number density of the 3 light neutrinos

Using the temperature $T_0(\nu_i) = (4/11)^{1/3} T_0 = (1.94537 \pm 0.00043)\,\mathrm{K}$ of the neutrino background [1-6, 9, 10], we obtain generally the present neutrino number densities (for every subscript $i = \mathrm{e}, \mu, \tau$) to

$$n_0(\nu_i) = \frac{3}{4} \frac{2.404113806}{\pi^2} \frac{(k[(4/11)^{1/3} T_0])^3}{(\hbar c)^3} = \\ = 112.016 \pm 0.074\ \mathrm{cm}^{-3}. \qquad (6.1)$$

They are necessary for the derivation of the neutrino density parameters of the universe in the next chapter.

7 The light neutrino density parameters of the universe for the two approximations

Analogously to the baryon number density of the present universe [1-6, 9, 10], for the present light neutrino number density (6.1), we can also assume the connection

$$n_0(\nu_i) = \Omega_\nu(\nu_i) \frac{\rho_{0C} c^2}{E_0(\nu_i)}, \qquad (7.1)$$

so that the neutrino density parameters of the universe are determined by

$$\Omega_\nu(\nu_i) = E_0(\nu_i) \frac{n_0(\nu_i)}{\rho_{0C} c^2}. \qquad (7.2)$$

Considering the radiation density parameter

$$\Omega_\gamma = \frac{\rho_0(\gamma)}{\rho_{0C}} = (2.4728 \pm 0.0025) \cdot 10^{-5} h^{-2}, \tag{7.3}$$

where the photon energy density $\rho_0(\gamma)c^2$ is given by

$$\rho_0(\gamma)\, c^2 = \frac{\pi^2}{15} \frac{(kT_0)^4}{(\hbar c)^3} = (0.26057 \pm 0.00023)\ \mathrm{eV\,cm^{-3}}, \tag{7.4}$$

we obtain

$$\Omega_\nu(\nu_i) = E_0(\nu_i) \frac{n_0(\nu_i)}{\rho_0(\gamma)\, c^2} \Omega_\gamma = \frac{45}{11} \frac{2.4041138}{\pi^4} \frac{E_0(\nu_i)}{kT_0} \Omega_\gamma =$$
$$= (429.889 \pm 0.095) \left[E_0(\nu_i)/\mathrm{eV}\right] \Omega_\gamma =$$
$$= (0.010630 \pm 0.000013) \left[E_0(\nu_i)/\mathrm{eV}\right] h^{-2}. \tag{7.5}$$

Then, using the results (3.6) to (3.9) as well as (4.6) to (4.9) in Eq. (7.5), we find the present light neutrino density parameters of the universe for the old approximation to

$$\Omega_\nu(\nu_\mathrm{e}) = (1.641 \pm 0.054) \cdot 10^{-5} h^{-2} = (3.61 \pm 0.17) \cdot 10^{-5}, \tag{7.6}$$

$$\Omega_\nu(\nu_\mu) = (9.37 \pm 0.13) \cdot 10^{-5} h^{-2} = (2.063 \pm 0.059) \cdot 10^{-4}, \tag{7.7}$$

$$\Omega_\nu(\nu_\tau) = (5.348 \pm 0.045) \cdot 10^{-4} h^{-2} = (1.177 \pm 0.027) \cdot 10^{-3} \tag{7.8}$$

and

$$\Omega_\nu = \sum\nolimits_i \Omega_\nu(\nu_i) =$$
$$= (6.448 \pm 0.063) \cdot 10^{-4} h^{-2} = (1.419 \pm 0.035) \cdot 10^{-3} \tag{7.9}$$

as well as for the new approximation to

$$\Omega_\nu(\nu_\mathrm{e}) = (8.02 \pm 0.21) \cdot 10^{-7} h^{-2} = (1.765 \pm 0.072) \cdot 10^{-6}, \tag{7.10}$$

$$\Omega_\nu(\nu_\mu) = (9.23 \pm 0.12) \cdot 10^{-5} h^{-2} = (2.032 \pm 0.057) \cdot 10^{-4}, \tag{7.11}$$

$$\Omega_\nu(\nu_\tau) = (5.345 \pm 0.045) \cdot 10^{-4} h^{-2} = (1.177 \pm 0.027) \cdot 10^{-3} \tag{7.12}$$

and

$$\Omega_\nu = \sum\nolimits_i \Omega_\nu(\nu_i) =$$
$$= (6.276 \pm 0.057) \cdot 10^{-4} h^{-2} = (1.382 \pm 0.033) \cdot 10^{-3}. \tag{7.13}$$

The values (7.9) and (7.13) lie near to the lower boundary of the observed range $0.003 > \Omega_\nu > 0.0012$ (see Table I).

8 Rest energies and number densities of the 3 heavy neutrinos

Because the light neutrinos form no luminous matter, their value Ω_ν [see Eqs. (7.9) and (7.13)] must be taken into account at the dark matter $\Omega_{\rm dm}$ in first approximation. Therefore, in accordance with Table I, we must consider the present cosmological particle density parameters as follows:

$$\Omega_{\rm b} = (0.02237 \pm 0.00015)\, h^{-2} = 0.0493 \pm 0.0006\,, \tag{8.1}$$

$$\Omega_{\rm dm} = (0.1200 \pm 0.0012)\, h^{-2} = 0.264 \pm 0.007\,, \tag{8.2}$$

$$\Omega_{\rm dm} + \Omega_\nu = (0.1206 \pm 0.0012)\, h^{-2} = 0.265 \pm 0.007\,, \tag{8.3}$$

$$\Omega_{\rm m} = \Omega_{\rm dm} + \Omega_\nu + \Omega_{\rm b} = (0.1430 \pm 0.0014)\, h^{-2} = 0.315 \pm 0.007 \tag{8.4}$$

and

$$\Omega_\Lambda = (0.3113 \pm 0.0031)\, h^{-2} = 0.685 \pm 0.007\,. \tag{8.5}$$

By the curvature $\Omega_{\rm tot} = 1$, derived in Refs. [2-5], for $1 = \Omega_{\rm m} + \Omega_\Lambda$, we have introduced $\Omega_\Lambda = (0.3113 \pm 0.0031)\, h^{-2}$ in Eq. (8.5).

These cosmological parameter values (8.1) to (8.5) agree excellently with their data in Table I, so that they are used always in this work.

Consequently, we have the 3 present independent cosmological particle density parameters $\Omega_{\rm b}$, $\Omega_{\rm dm}$ and Ω_Λ, which are used to estimate the rest energies of the 3 heavy neutrinos ($\widetilde{\nu}_{\rm e} = \widetilde{\nu}_{\rm b}$, $\widetilde{\nu}_\mu = \widetilde{\nu}_{\rm dm}$ and $\widetilde{\nu}_\tau = \widetilde{\nu}_\Lambda$) in accordance with the results of Refs. [2-6] via the constant

$$\mathrm{A} = (1.0630 \pm 0.0013) \cdot 10^{-2}\, h^{-2}\ \mathrm{eV}^{-1} \quad (\text{see Eq. } (7.5))\,. \tag{8.6}$$

Then, in connection with the results of Refs. [2-6], the rest energies of the 3 heavy neutrinos are given by

$$E_0(\widetilde{\nu}_{\rm e}) = \frac{\Omega_{\rm b}}{\mathrm{A}} \cong 2.104 \pm 0.017\ \mathrm{eV}\,, \tag{8.7}$$

$$E_0(\tilde{\nu}_\mu) = \frac{\Omega_{\mathrm{dm}}}{\mathrm{A}} \cong 11.29 \pm 0.13\ \mathrm{eV} \tag{8.8}$$

and

$$E_0(\tilde{\nu}_\tau) = \frac{\Omega_\Lambda}{\mathrm{A}} \cong 29.29 \pm 0.33\ \mathrm{eV}. \tag{8.9}$$

Consequently, these present particle density parameters are proportional to the rest energy of the heavy neutrinos, so that they must be valid also for the correctly treated anti-universe [4, 6, 7] because the rest energies of their antiparticles possess correspondingly the same values.

Trivially, the present number densities of these 3 heavy neutrinos are determined by

$$n_0(\tilde{\nu}_\mathrm{e}) = \Omega_\mathrm{b} \frac{\rho_{0\mathrm{C}} c^2}{E_0(\tilde{\nu}_\mathrm{e})} \cong 112.0\ \mathrm{cm}^{-3}, \tag{8.10}$$

$$n_0(\tilde{\nu}_\mu) = \Omega_{\mathrm{dm}} \frac{\rho_{0\mathrm{C}} c^2}{E_0(\tilde{\nu}_\mu)} \cong 112.0\ \mathrm{cm}^{-3} \tag{8.11}$$

and

$$n_0(\tilde{\nu}_\tau) = \Omega_\Lambda \frac{\rho_{0\mathrm{C}} c^2}{E_0(\tilde{\nu}_\tau)} \cong 112.0\ \mathrm{cm}^{-3}, \tag{8.12}$$

i.e. they are identical with the present number densities (6.1) of the 3 light neutrinos. The neutrino densities (8.10) to (8.12) are a strong argument for the assumption that these 3 new neutrinos could be existent uniquely as heavy neutrinos. Because of this assumed existence of only 3 heavy neutrinos, the introduction of a fourth heavy neutrino of Refs. [2-6] is incorrect.

9 The properties of the 3 sterile neutrinos

In connection with the results of Refs. [2-6], we assume the probable existence of the 3 sterile neutrinos ($\hat{\nu}_\mathrm{e} = \hat{\nu}_\mathrm{b}$, $\hat{\nu}_\mu = \hat{\nu}_{\mathrm{dm}}$ and $\hat{\nu}_\tau = \hat{\nu}_\Lambda$), i.e. they must also couple with the 3 present independent cosmological particle density parameters Ω_b, Ω_{dm} and Ω_Λ. This coupling is possible, since the

lifetime of the sterile neutrinos, to which we will return in Eq. (13.14), is significantly greater than the age of the present universe

Our assumption is based on the reliable detection [28-33] of a weak unidentified X-ray emission line in the spectrum of galaxy clusters. For example, according to Bulbul et al. [28], this X-ray line is characterized by the energy

$$E = 3560 \pm 30\,\mathrm{eV} \tag{9.1}$$

and the photon flux

$$\Phi = (4.0^{+0.8}_{-0.8}[^{+1.8}_{-1.2}]) \cdot 10^{-6}\ \mathrm{cm}^{-2}\ \mathrm{s}^{-1} \tag{9.2}$$

at 68% [90%] confidence levels in a redshift range $0.35 \geq z \geq 0.01$. As reason of this X-ray line, in Ref. [28], a dark matter interpretation was assumed via the photon decay of the sterile neutrinos (long-sought dark matter particle candidates). This assumption is probable because they decay via the gravitation in one photon with the energy of their half rest energy and possess a lifetime larger than the age of the universe.

Consequently, because of the decay properties of these sterile neutrinos ($\hat{\nu}_\mu = \hat{\nu}_{\mathrm{dm}}$), we assume a rest energy to

$$E_0(\hat{\nu}_\mu) = 7120 \pm 60\,\mathrm{eV}, \tag{9.3}$$

i.e. the present number density can be found to

$$n_0(\hat{\nu}_\mu) = \Omega_{\mathrm{dm}} \frac{\rho_{0\mathrm{C}}\, c^2}{E_0(\hat{\nu}_\mu)} = 0.1776 \pm 0.0033\ \mathrm{cm}^{-3}. \tag{9.4}$$

Then, analogous to the results (8.10) to (8.12), we can assume the same number density for the remaining sterile neutrinos, i.e. we have also

$$n_0(\hat{\nu}_\mathrm{e}) = \Omega_\mathrm{b} \frac{\rho_{0\mathrm{C}}\, c^2}{E_0(\hat{\nu}_\mathrm{e})} = 0.1776 \pm 0.033\ \mathrm{cm}^{-3} \tag{9.5}$$

and

$$n_0(\hat{\nu}_\tau) = \Omega_\Lambda \frac{\rho_{0\mathrm{C}}\, c^2}{E_0(\hat{\nu}_\tau)} = 0.1776 \pm 0.033\ \mathrm{cm}^{-3}, \tag{9.6}$$

so that their rest energies are given by

$$E_0(\hat{\nu}_e) = \Omega_b \frac{\rho_{0C}\, c^2}{n_0(\hat{\nu}_e)} = 1327 \pm 25 \text{ eV} \tag{9.7}$$

and

$$E_0(\hat{\nu}_\tau) = \Omega_\Lambda \frac{\rho_{0C}\, c^2}{n_0(\hat{\nu}_\tau)} = 18470 \pm 343 \text{ eV}. \tag{9.8}$$

Using the results (8.10) to (8.12) as well as (9.4) to (9.6), we can form the constant ratios

$$\frac{n_0(\tilde{\nu}_e)}{n_0(\hat{\nu}_e)} = \frac{E_0(\hat{\nu}_e)}{E_0(\tilde{\nu}_e)} = \frac{n_0(\tilde{\nu}_\mu)}{n_0(\hat{\nu}_\mu)} = \frac{E_0(\hat{\nu}_\mu)}{E_0(\tilde{\nu}_\mu)} =$$

$$= \frac{n_0(\tilde{\nu}_\tau)}{n_0(\hat{\nu}_\tau)} = \frac{E_0(\hat{\nu}_\tau)}{E_0(\tilde{\nu}_\tau)} = 630.7 \pm 11.7. \tag{9.9}$$

Thus, we can introduce the general constant

$$\mathrm{B} = \frac{n_0(\hat{\nu}_i)}{n_0(\tilde{\nu}_i)} \mathrm{A} = 1.6854 \cdot 10^{-5}\, h^{-2} \text{ eV}^{-1} \quad (\text{see Eq. (8.6)}), \tag{9.10}$$

so that we see again our assumption of the realistic coupling between the 3 present independent particle density parameters (Ω_b, Ω_{dm} and Ω_Λ) and the rest energy of the sterile neutrinos to

$$E_0(\hat{\nu}_e) = \frac{\Omega_b}{\mathrm{B}} \cong 1327 \text{ eV}, \tag{9.11}$$

$$E_0(\hat{\nu}_\mu) = \frac{\Omega_{dm}}{\mathrm{B}} \cong 7120 \text{ eV} \tag{9.12}$$

and

$$E_0(\hat{\nu}_\tau) = \frac{\Omega_\Lambda}{\mathrm{B}} \cong 18470 \text{ eV}. \tag{9.13}$$

Taking the results (9.4) to (9.6) and introducing the decay energy photon fluxes $\phi(\hat{\nu}_i) = \frac{1}{2} E_0(\hat{\nu}_i)\, \Phi(\hat{\nu}_i)$, we obtain the ratios

$$\frac{E_0(\hat{\nu}_\mu)}{E_0(\hat{\nu}_e)} = \frac{\Phi(\hat{\nu}_e)}{\Phi(\hat{\nu}_\mu)} = \frac{\Omega_{dm}}{\Omega_b} = 5.364 \pm 0.089, \tag{9.14}$$

$$\frac{E_0(\hat{\nu}_\tau)}{E_0(\hat{\nu}_e)} = \frac{\Phi(\hat{\nu}_e)}{\Phi(\hat{\nu}_\tau)} = \frac{\Omega_\Lambda}{\Omega_b} = 13.92 \pm 0.23 \tag{9.15}$$

and

$$\frac{E_0(\hat{\nu}_\tau)}{E_0(\hat{\nu}_\mu)} = \frac{\Phi(\hat{\nu}_\mu)}{\Phi(\hat{\nu}_\tau)} = \frac{\Omega_\Lambda}{\Omega_{dm}} = 2.594 \pm 0.052 . \tag{9.16}$$

These decay photon fluxes $\Phi(\hat{\nu}_i)$ of the probable 3 sterile neutrinos can be estimated in chapter 12. By this assumed existence of only 3 sterile neutrinos, the introduction of a fourth sterile neutrino is incorrect in Refs. [2-6].

If the upper conclusions (9.3) to (9.8) of the 3 sterile neutrinos are true, they must be attributable to a more physical principle. Indeed, these conclusions are confirmed excellently by the ratios (9.9), since they permit the reasonable and more realistic assumption

$$\alpha_{GUT} \to \alpha_{GUT}(\hat{\nu}_i, \tilde{\nu}_i) = \left\{\frac{n_0(\hat{\nu}_i)}{n_0(\tilde{\nu}_i)}\right\}^{1/2} = \left\{\frac{E_0(\tilde{\nu}_i)}{E_0(\hat{\nu}_i)}\right\}^{1/2} =$$

$$= \left\{\frac{1}{630.7 \pm 11.7}\right\}^{1/2} = 0.03982 \pm 0.00037 \tag{9.17}$$

because of the agreement with Eq. (5.19) and the known value $\alpha_{GUT} \cong 0.04$.

This expression (9.17) can be write also in the form of

$$\alpha_{GUT}(\hat{\nu}_i, \tilde{\nu}_i)\, n_0(\tilde{\nu}_i) = \frac{n_0(\hat{\nu}_i)}{\alpha_{GUT}(\hat{\nu}_i, \tilde{\nu}_i)} \tag{9.18}$$

or

$$\alpha_{GUT}(\hat{\nu}_i, \tilde{\nu}_i)\, E_0(\hat{\nu}_i) = \frac{E_0(\tilde{\nu}_i)}{\alpha_{GUT}(\hat{\nu}_i, \tilde{\nu}_i)} . \tag{9.19}$$

Assuming $E_0(\mathrm{M}) = (6.844 \pm 0.028) \cdot 10^{17}$ GeV [see Eq. (5.15)] as a correct value and applying $\alpha_{GUT} = E_0(\mathrm{X})/E_0(\mathrm{M}) \to \alpha_{GUT}(\hat{\nu}_i, \tilde{\nu}_i) = E_0^{\hat{\nu}_i, \tilde{\nu}_i}(\mathrm{X})/E_0(\mathrm{M})$ [see Eq. 5.1)], we can derive a better value

$$E_0(\mathrm{X}) \to E_0^{\hat{\nu}_i, \tilde{\nu}_i}(\mathrm{X}) = (2.725 \pm 0.021) \cdot 10^{16}\ \mathrm{GeV} \tag{9.20}$$

than in Eq. (5.8), so that we can introduce the result (9.19) in the more plausible form of

$$\frac{E_0^{\hat{\nu}_i,\tilde{\nu}_i}(\mathrm{X})}{E_0(\mathrm{M})}E_0(\hat{\nu}_i)=\frac{E_0(\mathrm{M})}{E_0^{\hat{\nu}_i,\tilde{\nu}_i}(\mathrm{X})}E_0(\tilde{\nu}_i)\,, \tag{9.21}$$

whereat the assumed neutrino constant $\mathrm{C}_\nu \cong \mathrm{C_e}$ of Eqs. (5.5) and (5.6) becomes now equivalent to $\mathrm{C_e} \to \mathrm{C}_\nu = 1.0085$. Consequently, the 3 sterile neutrinos could be existent as one's own species.

These results are supported again by a solution, which is independent on the observations of Bulbul et al. if we take into account that the decay properties of the 3 sterile neutrinos (see above as well as chapters 12 to 14) permit only an effective existence of $^3\!/_2$ sterile neutrinos, so that by addition of the 3 light and 3 heave stable neutrinos we have totally (tot) $^{15}\!/_2$ neutrinos. Therefore, for the neutrino constant C_ν of Eq. (5.5), we can assume

$$\mathrm{C}_\nu \to \mathrm{C}_\nu^{\mathrm{tot}} \cong \mathrm{C_e}^{15/2} \cong 1.00116^{15/2} \cong 1.00873\,, \tag{9.22}$$

so that Eq. (5.6) yields

$$E_0(\mathrm{X}) \to E_0^{\mathrm{tot}}(\mathrm{X}) \cong \mathrm{C_e}^{15/2}\frac{E_0(\nu_i)}{\sum_i E_0(\nu_i)}\frac{E_0(\mathrm{e})}{\mu_\nu(\nu_i)}\mu_B =$$

$$\cong \mathrm{C_e}^{15/2}\frac{E_0(\mathrm{e})}{\sum_i E_0(\nu_i)}\frac{\mathrm{eV}}{3.203\cdot 10^{-19}} \cong (2.726 \pm 0.021)\cdot 10^{16}\,\mathrm{GeV}\,. \tag{9.23}$$

Thus, independently on the observations of Bulbul et al., by Eqs. (5.1) and (9.23), we obtain finally

$$\alpha_{\mathrm{GUT}} \to \alpha_{\mathrm{GUT}}^{\mathrm{tot}} = \frac{E_0^{\mathrm{tot}}(\mathrm{X})}{E_0(\mathrm{M})} = 0.03983 \pm 0.00047\,. \tag{9.24}$$

Using the assumptions (9.17) or (9.18), via Eq. (9.24), we get thus the independent final results

$$n_0(\hat{\nu}_i) = \alpha_{\mathrm{GUT}}^2\, n_0(\tilde{\nu}_i) \to$$

$$\to n_0^{\mathrm{tot}}(\hat{\nu}_i) = (\alpha_{\mathrm{GUT}}^{\mathrm{tot}})^2\, n_0(\tilde{\nu}_i) = (0.1777 \pm 0.0042)\,\mathrm{cm}^{-3} \tag{9.25}$$

and

$$E_0(\hat{\nu}_i) \to E_0^{\mathrm{tot}}(\hat{\nu}_i) = \frac{n_0(\tilde{\nu}_i)}{n_0^{\mathrm{tot}}(\hat{\nu}_i)}E_0(\tilde{\nu}_i) \tag{9.26}$$

in excellent agreement with Eqs. (9.4) to (9.9).

With that, we have proved that these 3 sterile neutrinos could be existent as one's own species via a more physical principle because of the improved SUSY GUT coupling constants $\alpha_{\mathrm{GUT}} \to \alpha_{\mathrm{GUT}}(\hat{\nu}_i, \widetilde{\nu}_i) = 0.03982 \pm 0.00037$ [see Eq. (9.17)] as well as $\alpha_{\mathrm{GUT}} \to \alpha_{\mathrm{GUT}}^{\mathrm{tot}} = 0.03983 \pm 0.00047$ [see Eq. (9.24)].

We will return to these 3 sterile neutrinos in chapters 13 and 14, where we complete their properties via the age of the final state of the universe, derived by their present number densities (9.4) to (9.6), which are confirmed excellently by the general result (9.25), i.e. we have used the neutrino number density $n_0(\hat{\nu}) = 0.1776\,\mathrm{cm}^{-3}$ of these probable 3 sterile neutrinos (as one's own species) in Eqs. (14.5) as well as (14.11) to (14.13). Thus, we have derived the basis of a new description of the complex neutrino problem in connection with the cosmology.

10 The Hubble constants

The considerations of this chapter are based on the works [1-10, 16-20].

In Refs. [2-4], we have generally derived $\mathrm{k} = 0$ by $\Omega_{\mathrm{tot}} = 1$ and $\Omega_{\mathrm{tot}}(z) = 1$, so that in the $\Lambda\mathrm{CDM}$ model the Friedmann-Lemaitre Equations are given by

$$H^2 = \left(\frac{\dot{R}}{R}\right)^2 = \frac{8\pi G_N}{3}\rho + \frac{c^2}{3}\Lambda \tag{10.1}$$

and

$$\frac{\ddot{R}}{R} = -\frac{4\pi G_N}{3}\left(\rho + \frac{3P}{c^2}\right) + \frac{c^2}{3}\Lambda\,, \tag{10.2}$$

where $H = H(t)$ is the Hubble parameter, $R = R(t)$ is the scale factor, ρ is the mean mass density, Λ is the cosmological constant and P is the isotropic pressure. For $\Lambda = 0$ and $z << z_{\mathrm{eq}}$ (z_{eq} see Table I), in the matter-dominated universe, according to Refs. [1-6, 9, 10], we must consider the mass density

$$\rho = \rho_{\mathrm{m}} + \rho_{\Lambda} = (\Omega_{\mathrm{m}}\left(\frac{R_0}{R}\right)^3 + \Omega_{\Lambda})\,\rho_{0\mathrm{C}}\,, \tag{10.3}$$

where ρ_{0C} describes the present critical mass density

$$\rho_{0C} = \frac{3H_0^2}{8\pi G_N}. \tag{10.4}$$

In this case, the Friedmann Equation (10.1) yields the solutions

$$t = \frac{2}{3H_0\,\Omega_\Lambda^{1/2}} \ln \frac{\sqrt{\Omega_\Lambda (1+z)^{-3}} + \sqrt{\Omega_\Lambda (1+z)^{-3} + \Omega_m}}{\sqrt{\Omega_m}} \tag{10.5}$$

and

$$\frac{R}{R_0} = \frac{1}{1+z} = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \left[\sinh(\tfrac{3}{2}\Omega_\Lambda^{1/2} H_0 t)\right]^{2/3} =$$

$$= \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} \left[\frac{e^{\frac{3}{2}\Omega_\Lambda^{1/2} H_0 t} - e^{-\frac{3}{2}\Omega_\Lambda^{1/2} H_0 t}}{2}\right]^{2/3}. \tag{10.6}$$

If $\rho_m << \rho_\Lambda$, Eq. (10.1) provides the "present accelerated" exponential expansion

$$R = R_0\, e^{\Omega_\Lambda^{1/2} H_0 (t - t_0)}, \tag{10.7}$$

where $t_0 = 4.3541 \cdot 10^{17}$ s (see Table I).

According to Refs. [4-6], the dark mass density $\rho_\Lambda = \Omega_\Lambda \rho_{0C}$ can be interpreted as the vacuum mass density

$$\rho_{vac} = p_\Lambda = \Omega_\Lambda \rho_{0C} = \frac{c^2 \Lambda}{8\pi G_N}, \tag{10.8}$$

so that we can introduce the negative pressure

$$P = -\rho_{vac}\, c^2 = -p_\Lambda c^2 = -\Omega_\Lambda \rho_{0C}\, c^2 = -\frac{c^4 \Lambda}{8\pi G_N}. \tag{10.9}$$

Then, using the expressions (10.3) as well as (10.8) and (10.9) in Eq. (10.2), we obtain

$$\frac{\ddot{R}}{R} = -\frac{4\pi G_N}{3}(\rho_m - 2\rho_{vac}) = -\frac{1}{2} H_0^2 \left[\Omega_m (R_0/R)^3 - 2\,\Omega_\Lambda\right]. \tag{10.10}$$

Assuming $\ddot{R}=0$ directly prior to the beginning of the "present accelerated (acc)" exponential expansion, Eq. (10.10) yields

$$1+z_{\mathrm{acc}}=\left(\frac{2\,\Omega_\Lambda}{\Omega_{\mathrm{m}}}\right)^{1/3}=1.633\pm 0.011\,, \tag{10.11}$$

i.e. the redshift condition (10.11) defines the normal beginning of the "present accelerated" exponential expansion.

Then, because of the result (5.17), the scale factor R_{acc} of the beginning of the "present accelerated" exponential expansion is given by

$$R_{\mathrm{acc}}=\frac{R_0}{1+z_{\mathrm{acc}}}=(8.40\pm 0.12)\cdot 10^{25}\ \mathrm{m}. \tag{10.12}$$

Consequently, for the result (10.11), Eq. (10.7) provides

$$t_0-t=\frac{1}{\Omega_\Lambda^{1/2}\,H_0}\ln(1+z_{\mathrm{acc}})=2.712\cdot 10^{17}\ \mathrm{s}\,. \tag{10.13}$$

Taking the expressions (10.5) and (10.11), for the beginning of the "present accelerated" exponential expansion, we obtain

$$t=\tilde{t}=\frac{2}{3\,H_0\,\Omega_\Lambda^{1/2}}\ln\frac{\sqrt{\Omega_\Lambda\,(1+z_{\mathrm{acc}})^{-3}}+\sqrt{\Omega_\Lambda\,(1+z_{\mathrm{acc}})^{-3}+\Omega_{\mathrm{m}}}}{\sqrt{\Omega_{\mathrm{m}}}}=$$

$$=2.428\cdot 10^{17}\ \mathrm{s}=7.69\ \mathrm{Gyr}\,, \tag{10.14}$$

so that Eq. (10.6) gives again

$$R=R_0\left(\frac{\Omega_{\mathrm{m}}}{\Omega_\Lambda}\right)^{1/3}\left[\frac{\mathrm{e}^{3/2\,\Omega_\Lambda^{1/2}H_0\tilde{t}}-\mathrm{e}^{-3/2\,\Omega_\Lambda^{1/2}H_0\tilde{t}}}{2}\right]^{2/3}=8.40\cdot 10^{25}\ \mathrm{m}\,. \tag{10.15}$$

Thus, analogous to Eq. (10.13), because of the results (10.14) and (10.15), we assume a provisional Hubble constant H^*_{acc}, so that we can also introduce

$$t_0-\tilde{t}=\frac{1}{H^*_{\mathrm{acc}}}\ln(1+z_{\mathrm{acc}})=1.926\cdot 10^{17}\ \mathrm{s}\,. \tag{10.16}$$

Then, for example, in the cases of the results (10.13) and (10.16), at identical values $1+z_{\mathrm{acc}}$, this provisional Hubble constant is determined by

$$H^*_{\mathrm{acc}} = \frac{t_0 - t}{t_0 - \tilde{t}}\,\Omega_\Lambda^{1/2} H_0 = 2.546 \cdot 10^{-18}\,\mathrm{s} = 78.56\,\mathrm{km\,s^{-1}\,Mpc^{-1}}. \qquad (10.17)$$

Considering more generally these provisional Hubble constants than $H^*_{\mathrm{acc}}(t,\tilde{t})$, then, for $z = z_{\mathrm{acc}} \to 0$, i.e. at $t = \tilde{t} = t_0$, the factor $(t_0 - t)/(t_0 - \tilde{t})$ is indeterminate, so that we must calculate its limiting value, which is determined by

$$\lim_{z\to 0} \frac{t_0 - t}{t_0 - \tilde{t}} = \frac{\sqrt{\Omega_\Lambda} + \sqrt{\Omega_\Lambda + \Omega_{\mathrm{m}}}}{\sqrt{\Omega_\Lambda} + \Omega_\Lambda / \sqrt{\Omega_\Lambda + \Omega_{\mathrm{m}}}} = 1.2083. \qquad (10.18)$$

Consequently, for Eq. (10.17), in the case (10.18), we have

$$H^*_{\mathrm{acc}} = \Omega_\Lambda^{1/2} H_0 \lim_{z\to 0} \frac{t_0 - t}{t_0 - \tilde{t}} =$$

$$= \frac{\sqrt{\Omega_\Lambda} + \sqrt{\Omega_\Lambda + \Omega_{\mathrm{m}}}}{\sqrt{\Omega_\Lambda} + \Omega_\Lambda / \sqrt{\Omega_\Lambda + \Omega_{\mathrm{m}}}}\,\Omega_\Lambda^{1/2} H_0 =$$

$$= H_0 = (2.184 \pm 0.016) \cdot 10^{-18}\,\mathrm{s^{-1}} = 67.4 \pm 0.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}, \qquad (10.19)$$

i.e. for $z = z_{\mathrm{acc}} = 0$ the provisional Hubble constant H^*_{acc} is identical with the present Hubble constant $H_0 = 67.4 \pm 0.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ of Planck 2013 (see Refs. 16 and 20 as well as Table I).

However, more generally, during the "present" exponential expansion, Eq. (10.10) must fulfill the condition

$$\frac{\ddot{R}}{H_0 R} = \Omega_\Lambda H_0 - \frac{1}{2}\Omega_{\mathrm{m}}\,(R_0/R)^3 H_0 \geq 0, \qquad (10.20)$$

so that we can assume

$$\frac{\Omega_\Lambda}{\Omega_{\mathrm{m}}} R^3 H_0 \geq \frac{1}{2} R_0^3 H_0\,. \qquad (10.21)$$

Then, instead of Eq. (10.6), we assume the scale factor

$$R_1(t) = \left(\frac{\Omega_{\mathrm{m}}}{\Omega_\Lambda}\right)^{1/3} R_0\,\sqrt{f_1(t)}\,, \qquad (10.22)$$

whereas for the also generally valid Eq. (10.15) we introduce the scale factor

$$R_2(\tilde{t}) = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} R_0\, f_2(\tilde{t}), \tag{10.23}$$

where $f_1(t)$ and $f_2(\tilde{t})$ describe time-dependent functions, i.e. we must form

$$R = \frac{R_1^2(t)}{R_2(\tilde{t})} = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} R_0\, \frac{f_1(t)}{f_2(\tilde{t})} = \left(\frac{\Omega_m}{\Omega_\Lambda}\right)^{1/3} R_0\, f(t,\tilde{t}), \tag{10.24}$$

whereat we have used $f(t,\tilde{t}) = f_1(t)/f_2(\tilde{t})$, so that instead of the result (10.21) we find

$$f^3(t,\tilde{t})\, H_0 \geq \frac{1}{2} H_0 . \tag{10.25}$$

Then, because of the results (10.17) to (10.19), we assume

$$f(t,\tilde{t}) = \frac{(t_0 - t)/(t_0 - \tilde{t})}{\lim_{z\to 0} (t_0 - t)/(t_0 - \tilde{t})}, \tag{10.26}$$

so that via the condition (10.25) in the result (10.17) we can introduce the definitive Hubble constant H_{acc} of the "present accelerated" exponential expansion [because of $H^*_{acc} = H_0$ in Eq. (10.19)] to

$$H_{acc} = f^3(t,\tilde{t})\, H^*_{acc} = \frac{t_0 - t}{t_0 - \tilde{t}}\, \Omega_\Lambda^{1/2} f^3(t,\tilde{t}) H_0 =$$

$$= \frac{\left[(t_0 - t)/(t_0 - \tilde{t})\right]^4}{\left[\lim_{z\to 0} (t_0 - t)/(t_0 - \tilde{t})\right]^3}\, \Omega_\Lambda^{1/2} H_0 \geq \frac{1}{2} H_0 . \tag{10.27}$$

Thus, for $z = 0$, the definitive Hubble constant H_{acc} is identical with the present Hubble constant $H_0 = 67.4 \pm 0.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ of Planck 2013 [see Eq. 10.19)]. Therefore, this Hubble constant $H_0 = 67.4 \pm 0.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ is precise and unique the Hubble constant of the present universe ($z = 0$), so that all hitherto existing considerations can be conserved for the universe via the Friedmann-Lemaitre Equations because H_0 must be considered as its present Hubble expansion rate, which is responsible among other things for the events in the different epochs of the universe.

In Table II, as a function of the identical redshift or blueshift conditions $1+z_{\mathrm{acc}}$, via the generally valid results (10.13) for t and (10.14) for $\tilde{t}$, several by Eq. (10.27) estimated values of the Hubble constants H_{acc} are compiled for the "present accelerated" exponential expansion.

Table II. The observed (H) and estimated (H_{acc}) values of the Hubble constant, using $t_0 = 4.3541 \cdot 10^{17}\,\mathrm{s}$ (see Table I).

$1+z_{\mathrm{acc}}$	H [a)] ($\mathrm{km\,s^{-1}Mpc^{-1}}$) ($0.01 < z < 0.15$)	H_{acc} ($\mathrm{km\,s^{-1}Mpc^{-1}}$)	$\tilde{t}$ ($10^{17}\,\mathrm{s}$)	$t_0 - t$ ($10^{17}\,\mathrm{s}$)	$t - t_0$ ($10^{17}\,\mathrm{s}$)
1.633		124.33±0.92	2.428	2.712	
1.5		109.53±0.81	2.710	2.242	
1.4		99.45±0.74	2.957	1.8608	
1.3		90.25±0.67	3.238	1.4510	
1.15		78.00±0.58	3.737	0.7729	
1.125	76.18±2.37	76.23±0.57	3.831	0.6514	
1.098	74.50±3.27	74.31±0.55	3.937	0.5170	
1.085	73.24±1.74	73.30±0.54	3.988	0.4513	
1.075	72.25±2.51	72.84±0.54	4.029	0.4000	
1.06	72.04±2.67	71.79±0.53	4.092	0.3222	
1.05		71.15±0.53	4.134	0.2698	
1.025		70.24±0.52	4.242	0.1366	
1		67.4 ±0.5	4.3541	0	

Table II. Continued.

$1+z_{acc}$	H [a] ($\mathrm{km\ s^{-1}Mpc^{-1}}$) ($0.01<z<0.15$)	H_{acc} ($\mathrm{km\ s^{-1}Mpc^{-1}}$)	$\tilde{t}$ (10^{17} s)	t_0-t (10^{17} s)	$t-t_0$ (10^{17} s)
0.95		63.75±0.47	4.592		0.2837
0.8		56.05±0.42	5.424		1.234
0.6		47.72±0.35	6.903		2.825
0.52		45.05±0.33	7.664		3.616
0.51		44.78±0.33	7.769		3.724
0.395		41.52±0.31	9.153		5.137
0.3		39.30±0.29	10.660		6.658
0.28		38.90±0.29	11.039		7.040
0.23		37.89±0.28	12.123		8.128
0.15		36.41±0.27	14.483		10.492
0.03		34.11±0.25	23.38		19.39
0.01929		33.75±0.25	25.83		21.83

[a] Typical values of Ref. [34] for the investigated redshift range.

For Table II, at the estimation of the Hubble constants (10.27), we have assumed that the corresponding present cosmological particle density parameters describe also the future evolution of the universe for the blueshift conditions $1+z_{acc}<1$.

The estimated values of the Hubble constant H_{acc}, which decrease from the highest value $H_{acc} = H_{max} = 124.33\,\mathrm{km\,s^{-1}Mpc^{-1}}$ at $1 + z_{acc} = 1.633$ to the lowest value $H_{acc} = H_{min} = 33.75\,\mathrm{km\,s^{-1}Mpc^{-1}}$ at $1 + z_{acc} = 0.01929$, show precisely and uniquely that the "present" exponential expansion (10.7) is decelerated and not accelerated, i.e. the approximation (10.27) is justified.

Then, for the final state of the universe, i.e. to the end of the decelerated "present" exponential expansion, defined by $H_{min} = 33.75\,\mathrm{km\,s^{-1}Mpc^{-1}}$ at $1 + z_{min} = 0.01929$, the assumption of a slow linear expansion is incorrect in Refs. [4-6]. This conclusion is supported uniquely and precisely by this new Hubble constant $H_{acc} = H_{min} = 33.75\,\mathrm{km\,s^{-1}Mpc^{-1}}$, since it fulfills the condition (10.25), so that we have here indeed the end of the "present" decelerated exponential expansion as the final state of the universe. The value $1 + z_{acc} = 1 + z_{min} = 0.01929$ results from the electron neutrino matter-radiation equality, which yields $1 + z_{min} = 1 + z_{new} = 2\,\Omega_\nu (\nu_e)_{new} / N(T)\,\Omega_\gamma$ [3-5] for this smallest massive particle with $\Omega_\nu (\nu_e)_{new} = (1.765 \pm 0.072) \cdot 10^{-6}$ [see Eq. (7.10)], where $N(T) \equiv 3.362244$ [9, 10] and $\Omega_\gamma = (5.44 \pm 0.08) \cdot 10^{-5}$ (see Table I). These values are used later to calculate the scale factor of the final state of the universe by Eq. (15.4) of the new approximation, so that our model of the complex neutrino problem for the particle physics and the cosmology is confirmed excellently by the value $1 + z_{new} = 1 + z_{acc} = 0.01929$.

In Table II, the estimated values H_{acc} are also compared with the Hubble constants H, determined exactly by Riess et al. [34] in the investigated redshift range $0.01 < z = z_{acc} < 0.15$. For this range, the estimated values H_{acc} agree excellently with these observed data.

Now, we determine exactly the physical reason for the new mean value $H = 73.24 \pm 1.74\,\mathrm{km\,s^{-1}Mpc^{-1}}$ of the Hubble constants, found by Riess et al. according to Table II. For this goal, instead of Eq. (10.8), via the functions $f(\{H_0 / H_{acc}\}^2)$ and $f(\Omega_\Lambda)$, we introduce the new vacuum mass density

$$\rho^{*}_{\mathrm{vac}} = f(\{H_0/H_{\mathrm{acc}}\}^2)\, f(\Omega_\Lambda)\ \rho_{0\mathrm{C}} = \frac{c^2\Lambda^{*}}{8\pi G_N}, \tag{10.28}$$

which is identical with Eq. (10.8) for $f(\{H_0/H_{\mathrm{acc}}\}^2)=1$ and $f(\Omega_\Lambda)=\Omega_\Lambda$.

Then, considering $f(\{H_0/H_{\mathrm{acc}}\}^2)=(H_0/H_{\mathrm{acc}})^2$ and $f(\Omega_\Lambda)=\Omega_\Lambda$, the expression (10.28) can be converted into the new vacuum mass density

$$\rho^{**}_{\mathrm{vac}} = (H_0/H_{\mathrm{acc}})^2\, \Omega_\Lambda\, \rho_{0\mathrm{C}} = \frac{c^2\Lambda^{**}}{8\pi G_N}, \tag{10.29}$$

so that we find its negative pressure to

$$P^{**} = -\rho^{**}_{\mathrm{vac}}\, c^2 = -\left(\frac{H_0}{H_{\mathrm{acc}}}\right)^2 \Omega_\Lambda\, \rho_{0\mathrm{C}}\, c^2 = -\frac{c^4\Lambda^{**}}{8\pi G_N}. \tag{10.30}$$

Consequently, instead of the result (10.10), because of $\rho_{\mathrm{vac}} \to \rho^{**}_{\mathrm{vac}}$ and $P \to P^{**}$, Eq. (10.2) is determined by

$$\frac{\ddot{R}}{R} = -\frac{1}{2}H_0^2\left[\Omega_{\mathrm{m}}(R_0/R)^3 - 2(H_0\,\Omega_\Lambda^{1/2}/H_{\mathrm{acc}})^2\right]. \tag{10.31}$$

Then, considering the Hubble constant $H_{\mathrm{acc}} = 124.33\,\mathrm{km\,s^{-1}Mpc^{-1}}$ relative to the normal beginning of the "present" decelerated exponential expansion at $1+z_{\mathrm{acc}} = 1.633$ in Eq. (10.11), the reduced new vacuum mass density (10.29) [in comparison with the result (10.8)] leads to a delayed beginning of the "present" decelerated exponential expansion with again $\ddot{R}=0$ before its start, so that Eq. (10.31) is fixed (fix) on the additional redshift condition

$$1+z_{\mathrm{fix}} = \left[\frac{2\,\Omega_\Lambda}{\Omega_{\mathrm{m}}}\left(\frac{H_0}{H_{\mathrm{acc}}}\right)^2\right]^{1/3} = 1.085 \pm 0.011. \tag{10.32}$$

Using Eq. (10.27), the redshift condition (10.32) yields the new present Hubble constant

$$H_{\mathrm{acc}} = 73.30 \pm 0.54\,\mathrm{km\,s^{-1}Mpc^{-1}}, \tag{10.33}$$

which is determined by a delayed beginning of the "present" decelerated exponential expansion as physical reason. It agrees well with the experimental values H of Riess et al. in Table II, whereat the value (10.33) represents

excellently the mean value of $H = 73.24 \pm 1.74\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, derived from the measured Hubble constants [34]. With that, we have explained plausibly the mystery to the observed values of the Hubble constant by the decelerated exponential expansion (see above).

Finally, we prove still uniquely and precisely that the final state of the universe, i.e. the end of the "present" decelerated exponential expansion, is defined by $H_{\mathrm{acc}} = H_{\mathrm{min}} = 33.75\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ at $1 + z_{\mathrm{acc}} = 1 + z_{\mathrm{min}} = 0.01929$ (see above or Table II).

For this goal, we use again $H^{*}_{\mathrm{acc}} = H_0$ of Eq. (10.19) in Eq. (10.25), so that we can write the result (10.25) according to Eqs. (10.20) and (10.21) via the expression (10.27) in the form of

$$\frac{\ddot{R}}{H_0} = f^3(t,\tilde{t})\,H_0\,R - \frac{1}{2}H_0\,R = f^3(t,\tilde{t})\,H^{*}_{\mathrm{acc}}\,R - \frac{1}{2}H_0\,R =$$

$$= H_{\mathrm{acc}}\,R - \frac{1}{2}H_0\,R = H_{\mathrm{min}}\,R_{\mathrm{new}} - \frac{1}{2}H_0\,R_{\mathrm{new}} = 0\,, \qquad (10.34)$$

i.e. the limits $H_{\mathrm{acc}} = H_{\mathrm{min}}$ and $R = R_{\mathrm{new}}$ are defined by $H_{\mathrm{min}} \cong \frac{1}{2}H_0$ and $R_{\mathrm{new}} = R_0/(1 + z_{\mathrm{min}})$, so that the condition (10.34) is fulfilled. Similarly, we can convert the "present" decelerated exponential (exp) expansion (10.7) into

$$R_{\mathrm{exp}} = R_0\,\mathrm{e}^{\{(t_0 - t)/(t_0 - \tilde{t})\}\Omega_\Lambda^{1/2}H_0(\tilde{t} - t_0)} = R_0\,\mathrm{e}^{H^{*}_{\mathrm{acc}}(\tilde{t} - t_0)} \rightarrow$$

$$\rightarrow R_0\,\mathrm{e}^{f^3(t,\tilde{t})\,H^{*}_{\mathrm{acc}}(\tilde{t} - t_0)} = R_0\,\mathrm{e}^{H_{\mathrm{acc}}(\tilde{t} - t_0)} = R_0\,\mathrm{e}^{H_{\mathrm{min}}(\tilde{t} - t_0)}\,, \qquad (10.35)$$

so that for $\ddot{R}_{\mathrm{exp}}/H_{\mathrm{min}}$ we get

$$\frac{\ddot{R}_{\mathrm{exp}}}{H_{\mathrm{min}}} = H_{\mathrm{min}}\,R_{\mathrm{exp}} \cong \frac{1}{2}H_0\,R_{\mathrm{exp}}\,. \qquad (10.36)$$

Then, introducing $\ddot{R}_{\mathrm{new}}/H_{\mathrm{min}} = H_{\mathrm{min}}\,R_{\mathrm{new}} = \frac{1}{2}H_0\,R_{\mathrm{new}}$ via Eq. (10.34) and using the result (10.36) for the final state of the universe, i.e. at the end of the "present" decelerated exponential expansion, we find indeed the result

$$\frac{\ddot{R}}{H_0} = \frac{\ddot{R}_{\mathrm{new}}}{H_{\mathrm{min}}} - \frac{\ddot{R}_{\mathrm{exp}}}{H_{\mathrm{min}}} = H_{\mathrm{min}}\,R_{\mathrm{new}} - \frac{1}{2}H_0\,R_{\mathrm{exp}} = 0 \qquad (10.37)$$

as expected for

$$R_{\exp} = R_{\text{new}} = R_0/(1+z_{\min}), \tag{10.38}$$

to which we will return at Eq. (15.4). An analogous result yields the direct consideration of the acceleration of Eq. (10.35), when we introduce

$$\ddot{R}_{\exp} = H_{\min}^2 R_{\exp} \cong \frac{1}{2} H_{\min} H_0 R_{\exp} \cong \frac{1}{4} H_0^2 R_{\exp} \cong$$

$$\cong \ddot{R}_{\text{new}} \cong \frac{1}{2} H_{\min} H_0 R_{\text{new}} \cong \frac{1}{4} H_0^2 R_{\text{new}}. \tag{10.39}$$

Consequently, considering Eq. (10.34) in the form of

$$\ddot{R} = H_{\min} H_0 R_{\text{new}} - \frac{1}{2} H_0^2 R_{\text{new}} = 0, \tag{10.40}$$

instead of Eqs. (10.38) to (10.40), in agreement with Eq. (10.37), we can write

$$\ddot{R} = 2\ddot{R}_{\exp} - 2\ddot{R}_{\text{new}} = H_{\min} H_0 R_{\exp} - \frac{1}{2} H_0^2 R_{\text{new}} = 0, \tag{10.41}$$

i.e. our expectation (10.38) is fulfilled for the final state of the universe as the end of the "present" decelerated exponential expansion. In Eq. (10.41), the double accelerates "$2\ddot{R}_{\exp}$" and "$2\ddot{R}_{\text{new}}$" are clear, since (in these two cases) Eqs. (10.1) and (10.2) yield the same results, which must be added up.

11 The mean negative acceleration (Pioneer anomaly), the astronomical unit changing and the flyby anomaly

In this chapter, in contrast to the Refs. [1-6], we derive more correctly the mean negative acceleration (Pioneer anomaly) [1-6] the astronomical unit changing [1-6] and the flyby anomaly [1, 3]. We begin with the estimation of the mean negative acceleration. For this goal, instead of Eq. (10.28), via $f(\{H_0/H_{\text{acc}}\}^2) = (H_{\text{acc}}/H_0)^2$ and $f(\Omega_\Lambda) = 1/\Omega_\Lambda$, we can assume as new vacuum mass density

$$\rho_{\text{vac}}^{***} = \left(\frac{H_{\text{acc}}}{\Omega_\Lambda^{1/2} H_0}\right)^2 \rho_{0C} = \frac{c^2 \Lambda^{***}}{8\pi G_N}, \tag{11.1}$$

so that we can also introduce the negative pressure

$$P^{***} = -\rho_{\mathrm{vac}}^{***} c^2 = -\left(\frac{H_{\mathrm{acc}}}{\Omega_\Lambda^{1/2} H_0}\right)^2 \rho_{0\mathrm{C}}\, c^2 = -\frac{c^4 \Lambda^{***}}{8\pi G_N}. \qquad (11.2)$$

Then, instead of the result (10.10), because of $\rho_{\mathrm{vac}} \to \rho_{\mathrm{vac}}^{***}$ and $P \to P^{***}$, Eq. (10.2) gives

$$\frac{\ddot{R}}{R} = -\frac{1}{2} H_0^2 \left[\Omega_{\mathrm{m}} (R_0/R)^3 - 2(H_{\mathrm{acc}}/\Omega_\Lambda^{1/2} H_0)^2\right], \qquad (11.3)$$

whereat we use again the Hubble constant $H_{\mathrm{acc}} = 124.33\,\mathrm{km\,s^{-1}Mpc^{-1}}$ relative to $\Omega_\Lambda^{1/2} H_0$ at $1 + z_{\mathrm{acc}} = 1.633$ of the normal beginning of the "present" decelerated exponential expansion in Eq. (10.11). Consequently, in comparison with the result (10.8), the increased new vacuum mass density (11.1) leads to a new physical event of a premature beginning of the "present" decelerated exponential expansion with again $\ddot{R} = 0$ before its start, so that Eq. (11.3) must define the new redshift condition

$$1 + z_{\mathrm{MNA}} = \left[\frac{2}{\Omega_{\mathrm{m}}}\left(\frac{H_{\mathrm{acc}}}{\Omega_\Lambda^{1/2} H_0}\right)^2\right]^{1/3} = 3.160 \pm 0.036, \qquad (11.4)$$

i.e. we can assume that the redshift condition (11.4) must be responsible for a mean negative acceleration of the universe at the scale factor

$$R_{\mathrm{MNA}} = \frac{R_0}{1 + z_{\mathrm{MNA}}} = (4.342 \pm 0.081) \cdot 10^{25}\,\mathrm{m}, \qquad (11.5)$$

since the result (10.10) yields a mean negative acceleration $\ddot{R}_{\mathrm{MNA}}$ to

$$\ddot{R} = -\frac{1}{2} H_0^2 \left[\Omega_{\mathrm{m}} (R_0/R)^3 - 2\,\Omega_\Lambda\right] R_{\mathrm{MNA}} =$$

$$= -\frac{1}{2} H_0 c \left[\Omega_{\mathrm{m}} (1 + z_{\mathrm{MNA}})^2 - 2\,\Omega_\Lambda / (1 + z_{\mathrm{MNA}})\right] =$$

$$= (-8.88 \pm 0.74) \cdot 10^{-8}\,\mathrm{cm\,s^{-2}}. \qquad (11.6)$$

This mean negative acceleration (11.6) agrees excellently with the observed [11] Pioneer (P) anomaly

$$\ddot{R}_{\mathrm{P}} = a_{\mathrm{P}} = (-8.74 \pm 1.33) \cdot 10^{-8}\,\mathrm{cm\,s^{-2}}, \tag{11.7}$$

which was analysed in detail by Ref. [12].

An astronomical unit changing was observed by two independent researcher groups, analysing radio echoes from the planets [13, 14]. According to Krasinsky and Brumberg [13], the astronomical unit change was found to

$$\frac{d}{dt}\mathrm{AU} = (15 \pm 4)\,\mathrm{cm\,yr^{-1}}, \tag{11.8}$$

where $\mathrm{AU} = 1.49597870700(3) \cdot 10^{11}\,\mathrm{m}$ represents the astronomical unit [17].

This astronomical unit changing can be explained correctly by Eq. (10.1) for $\Lambda = 0$ via the mass density (10.3), using the results (11.4) and (11.5), if we assume

$$\frac{d}{dt}\mathrm{AU} = \dot{R}_{\mathrm{MNA}}\ \mathrm{corr(AU)} =$$

$$= H_0 \sqrt{\Omega_{\mathrm{m}} (R_0 / R_{\mathrm{MNA}})^3 + \Omega_\Lambda}\ R_{\mathrm{MNA}}\ \mathrm{corr(AU)}, \tag{11.9}$$

where the correction constant corr(AU) is necessary as a proportionality factor, since the radio echoes, which are observed on the surface of the earth, are reflected on the surface of the planets.

Thus, the radio signals run twice the distance $\Delta\mathrm{AU} = \mathrm{AU} - (R_{\mathrm{earth}} + R_{\mathrm{sun}})$, so that this correct constant must be assumed as

$$\mathrm{corr(AU)} = 2\frac{\mathrm{AU} - \Delta\mathrm{AU}}{R_{\mathrm{acc}}} = 2\frac{R_{\mathrm{earth}} + R_{\mathrm{sun}}}{R_{\mathrm{acc}}}, \tag{11.10}$$

where $R_{\mathrm{earth}} = 6.3781 \cdot 10^6\,\mathrm{m}$ and $R_{\mathrm{sun}} = 6.957 \cdot 10^8\,\mathrm{m}$ are the exact nominal equatorial radii of the earth and the sun [20]. More generally, the denominator of this corr(AU) must be defined by the scale factor R_{acc} of Eq. (10.12), since this value determines the normal beginning of the decelerated exponential expansion.

Thus, by Eqs. (11.9) and (11.10), we find the (mean) astronomical unit changing to

$$\frac{d}{dt}\mathrm{AU} = 2c\sqrt{\Omega_{\mathrm{m}}(1+z_{\mathrm{MNA}})+\Omega_{\Lambda}/(1+z_{\mathrm{MNA}})^2}\;\frac{R_{\mathrm{earth}}+R_{\mathrm{sun}}}{R_{\mathrm{acc}}} =$$

$$= (5.17 \pm 0.21)\cdot 10^{-7}\,\mathrm{cm\,s^{-1}} = (16.31 \pm 0.68)\ \mathrm{cm\,yr^{-1}}. \qquad (11.11)$$

The theoretical value (11.11) agrees excellently with the result (11.8) of the analysis of all available radiometric measurements in the time span 1961-2003 [13].

This result is in accordance with a later analysis of the solar-system anomalies in Ref. [15], which for the astronomical unit changing permits the limiting conditions $0 \le {}^{d}\!/_{dt}\,\mathrm{AU} \le 20\,\mathrm{cm\,yr^{-1}}$.

With that we have plausible explained the mystery of the astronomical unit changing.

Now, we evaluate the flyby anomaly.

For this goal, we assume that as real physical effect the mean negative acceleration $\ddot{R}_{\mathrm{MNA}} = -8.88\cdot 10^{-8}\,\mathrm{cm\,s^{-2}}$ [see Eq. (11.6)] can explain the observed anomalous small orbital-velocity differences ΔV_{∞}, denoted as flyby anomaly [35].

This flyby anomaly observed during the Earth flybys by the Galileo, NEAR, Cassini, Rosetta and MESSENGER spacecraft, which were launched between December 1990 and September 2005 to Jupiter, to an asteroid, to a comet, to Saturn and Mercury, respectively.

For these spacecraft's, at the closest approach, the most important Earth flyby parameters, tabulated in Ref. [35], are given in Table III.

Thus, we perform the following considerations.

The Earth (E) rotation period is

$$t_{\mathrm{E}} \cong 24\,\mathrm{h} \cong 8.64\cdot 10^{4}\,\mathrm{s}. \qquad (11.12)$$

Because of the exact nominal Earth equatorial radius $R_{\mathrm{Earth}} = 6.3781\cdot 10^{6}\,\mathrm{m}$ (see above), we get the Earth circular orbit velocity

$$V_{\mathrm{E}} \cong \frac{2\pi}{t_{\mathrm{E}}} R_{\mathrm{Earth}} = 463.83\,\mathrm{m\,s^{-1}}. \qquad (11.13)$$

However, by this large Earth circular orbit velocity, we cannot explain directly the flyby anomaly.

Therefore, we can assume only still that the Pioneer anomaly a_{P} [see Eq. (11.7)], which must agree with the real physical effect of the negative acceleration $\ddot{R}_{\mathrm{MNA}}$, has an influence on the spacecraft's.

TABLE III. The most important Earth flyby parameters at the closest approach for Galileo, NEAR, Cassini, Rosetta and MESSENGER spacecraft (see text).

Parameter	**Galileo-I**	**Galileo-II**	**NEAR**
V_∞ [km/s]	8.949	8.877	6.851
δ_{i} [deg]	-12.52	-34.26	-20.76
δ_{o} [deg]	-34.15	-4.87	-71.96
ΔV_∞ [mm/s]	3.92 ± 0.3	-4.6 ± 1.0	13.46 ± 0.01
ΔV_{FB} [mm/s]	4.24	-4.77	10.46

TABLE III. Continued.

Parameter	**Cassini**	**Rosetta**	**MESSENGER**
V_∞ [km/s]	16.010	3.863	4.056
δ_{i} [deg]	-12.92	-2.81	31.44
δ_{o} [deg]	-4.99	-34.29	-31.92
ΔV_∞ [mm/s]	-2 ± 1	1.80 ± 0.03	0.02 ± 0.01
ΔV_{FB} [mm/s]	-1.968	0.919	0.0258

For the flyby (FB) anomaly of the spacecraft's, we assume reasonably as a result of their orbit parameters at the closest approach that their "changed (C) rotation periods t_{CFB}" are given approximately by

$$t_{\mathrm{CFB}} \approx t_{\mathrm{E}} \left(\frac{V_\infty}{V_{\mathrm{E}}} \right)^2 (\cos\delta_{\mathrm{o}} - \cos\delta_{\mathrm{i}}), \tag{11.14}$$

where V_∞ is the osculating hyperbolic excess spacecraft velocity, whereas δ_{i} and δ_{o} represent the right declination of the incoming (i) and outgoing (o) osculating asymptotic spacecraft velocity vectors, respectively.

In this result (11.14), the square ratio $(V_\infty / V_{\mathrm{E}})^2$ of the spacecraft's is equivalent to the relative changing of their kinetic energies.

Thus, the result (11.14), multiplied by the mean negative acceleration $\ddot{R}_{\mathrm{MNA}} = -8.88 \cdot 10^{-8}\ \mathrm{cm\,s^{-2}}$, must yield semi-empirically the theoretical flyby velocity differences ΔV_{FB}, which are also summarized in Table III, as follows

$$\Delta V_{\mathrm{FB}} \approx \ddot{R}_{\mathrm{MNA}}\, t_{\mathrm{CFB}} \approx \ddot{R}_{\mathrm{MNA}}\, t_{\mathrm{E}} \left(\frac{V_\infty}{V_{\mathrm{E}}} \right)^2 (\cos\delta_{\mathrm{o}} - \cos\delta_{\mathrm{i}}). \tag{11.15}$$

According to Table III, the comparison of the experimental values ΔV_∞ as well as the theoretical values of our simple estimation ΔV_{FB} [see Eq. (11.15)] yields only a maximum deviation of a factor 2 at the Rosetta spacecraft. This satisfactory agreement of the calculated and observed flyby data confirms our assumption of the Pioneer anomaly as a real physical effect because of its excellent agreement with the mean negative acceleration.

Because the thermal recoil pressure computer simulation [36], which represents the Pioneer anomaly as an unreal physical effect, cannot describe the flyby anomaly, our doubts about the correctness of this computer simulation are justified.

Therefore, the successful evaluation of the mean negative acceleration, the astronomical unit changing and the flyby anomaly supports the assumptions of Sec. 10.

Consequently, this last conclusion proves uniquely and precisely that the mean negative acceleration (Pioneer anomaly) must exist as a real physical effect.

12 The photon densities of the sterile neutrinos

In this chapter, in contrast to works [1-5], we estimate correctly the probable decay photon fluxes of the sterile neutrinos. In this connection, we show that these 3 decay photon fluxes $\Phi(\hat{\nu}_e)$, $\Phi(\hat{\nu}_\mu)$ and $\Phi(\hat{\nu}_\tau)$ of the probable 3 sterile neutrinos (see Sec. 9) are uniquely and precisely a function of $1+z_{\mathrm{MNA}}$ and the 3 present independent cosmological particle density parameters Ω_b, Ω_{dm} and Ω_Λ, including the analogous neutrino density parameter Ω_ν.

Using the result $\Omega_m = \Omega_b + \Omega_{dm} + \Omega_\nu$ [see Eq. (8.4)], for our goal, the expression (11.11) is converted into

$$\left(\frac{1}{2c}\frac{R_{\mathrm{acc}}}{R_{\mathrm{earth}}+R_{\mathrm{sun}}}\frac{d}{dt}\mathrm{AU}\right)^2 = \Omega_m(1+z_{\mathrm{MNA}}) + \Omega_\Lambda/(1+z_{\mathrm{MNA}})^2 =$$

$$= (\Omega_b + \Omega_{dm} + \Omega_\nu)(1+z_{\mathrm{MNA}}) + \Omega_\Lambda/(1+z_{\mathrm{MNA}})^2 \,. \qquad (12.1)$$

By the left and right hand term of Eq. (12.1), we can form the corresponding expressions

$$\frac{R_{\mathrm{acc}}}{R_{\mathrm{earth}}+R_{\mathrm{sun}}}\left(\frac{1}{2c}\frac{d}{dt}\mathrm{AU}\right)^2 -$$

$$-\left[(\Omega_{dm}+\Omega_\nu)(1+z_{\mathrm{MNA}}) + \Omega_\Lambda/(1+z_{\mathrm{MNA}})^2\right]\frac{R_{\mathrm{earth}}+R_{\mathrm{sun}}}{R_{\mathrm{acc}}} =$$

$$= \Omega_b\,(1+z_{\mathrm{MNA}})\,\frac{R_{\mathrm{earth}}+R_{\mathrm{sun}}}{R_{\mathrm{acc}}}, \qquad (12.2)$$

$$\frac{R_{\rm acc}}{R_{\rm earth}+R_{\rm sun}}\left(\frac{1}{2c}\frac{d}{dt}{\rm AU}\right)^2 -$$

$$-\left[(\Omega_{\rm b}+\Omega_{\nu})(1+z_{\rm MNA})+\Omega_{\Lambda}/(1+z_{\rm MNA})^2\right]\frac{R_{\rm earth}+R_{\rm sun}}{R_{\rm acc}} =$$

$$=\Omega_{\rm dm}(1+z_{\rm MNA})\frac{R_{\rm earth}+R_{\rm sun}}{R_{\rm acc}}, \qquad (12.3)$$

$$2\left[(\Omega_{\rm b}+\Omega_{\rm dm})(1+z_{\rm MNA})+\Omega_{\Lambda}/(1+z_{\rm MNA})^2\right]\frac{R_{\rm earth}+R_{\rm sun}}{R_{\rm acc}} -$$

$$-\frac{1}{2}\frac{R_{\rm acc}}{R_{\rm earth}+R_{\rm sun}}\left(\frac{1}{c}\frac{d}{dt}{\rm AU}\right)^2 = -2\Omega_{\nu}(1+z_{\rm MNA})\frac{R_{\rm earth}+R_{\rm sun}}{R_{\rm acc}} \qquad (12.4)$$

and

$$\frac{R_{\rm acc}}{R_{\rm earth}+R_{\rm sun}}\left(\frac{1}{2c}\frac{d}{dt}{\rm AU}\right)^2 -$$

$$-\left[(\Omega_{\rm b}+\Omega_{\rm dm}+\Omega_{\nu})(1+z_{\rm MNA})\right]\frac{R_{\rm earth}+R_{\rm sun}}{R_{\rm acc}} =$$

$$=\left[\Omega_{\Lambda}/(1+z_{\rm MNA})^2\right]\frac{R_{\rm earth}+R_{\rm sun}}{R_{\rm acc}}. \qquad (12.5)$$

Via the photon (γ) number density

$$n_{\gamma}(\hat{\nu}_i)=\frac{\Phi(\hat{\nu}_i)}{c}, \qquad (12.6)$$

we can introduce the dimensionless ratio

$$\frac{n_{\gamma}(\hat{\nu}_i)}{n_0(\tilde{\nu}_i)}=\frac{\Phi(\hat{\nu}_i)}{n_0(\tilde{\nu}_i)c}, \qquad (12.7)$$

where $n_0(\tilde{\nu}_i)=112.016\,{\rm cm}^{-3}$ [see Eqs. (8.10) to (8.12)].

By the result (12.7), via the central term of Eq. (12.1), we can form the connection

$$\left(\frac{1}{2c}\frac{R_{\rm acc}}{R_{\rm earth}+R_{\rm sun}}\frac{d}{dt}{\rm AU}\right)^2\frac{n_\gamma(\hat{\nu}_i)}{n_0(\tilde{\nu}_i)}=$$

$$=\left[\Omega_{\rm m}(1+z_{\rm MNA})+\Omega_\Lambda/(1+z_{\rm MNA})^2\right]\frac{\Phi(\hat{\nu}_i)}{n_0(\tilde{\nu}_i)\,c}. \qquad (12.8)$$

Then, combining the results (12.2) to (12.5) with the corresponding expressions (12.8) for $i={\rm e},\mu$ or τ, we find the decay photon fluxes of the probable 3 sterile neutrinos as follows:

$$\phi(\hat{\nu}_\mu)=\frac{\Omega_{\rm b}\,(1+z_{\rm MNA})}{\Omega_{\rm m}(1+z_{\rm MNA})+\Omega_\Lambda/(1+z_{\rm MNA})^2}\frac{R_{\rm earth}+R_{\rm sun}}{R_{\rm acc}}n_0(\tilde{\nu}_\mu)\,c=$$

$$=(4.11\pm0.19)\cdot10^{-6}\ {\rm cm}^{-2}{\rm s}^{-1}, \qquad (12.9)$$

$$\phi(\hat{\nu}_{\rm e})=\frac{\Omega_{\rm dm}\,(1+z_{\rm MNA})}{\Omega_{\rm m}(1+z_{\rm MNA})+\Omega_\Lambda/(1+z_{\rm MNA})^2}\frac{R_{\rm earth}+R_{\rm sun}}{R_{\rm acc}}n_0(\tilde{\nu}_{\rm e})\,c=$$

$$=(2.20\pm0.10)\cdot10^{-5}\ {\rm cm}^{-2}{\rm s}^{-1} \qquad (12.10)$$

and

$$\phi(\hat{\nu}_\tau)=\frac{\left\{\Omega_\Lambda/(1+z_{\rm MNA})^2\right\}-2\Omega_\nu\,(1+z_{\rm MNA})}{\Omega_{\rm m}(1+z_{\rm MNA})+\Omega_\Lambda/(1+z_{\rm MNA})^2}\frac{R_{\rm earth}+R_{\rm sun}}{R_{\rm acc}}n_0(\tilde{\nu}_\tau)\,c=$$

$$=(1.581\pm0.073)\cdot10^{-6}\ {\rm cm}^{-2}{\rm s}^{-1}. \qquad (12.11)$$

We have used $\Omega_\nu=(6.276\pm0.057)\cdot10^{-4}\,h^{-2}=(1.382\pm0.033)\cdot10^{-3}$ [see Eq. (7.13)], since the new approximation is more realistic according to Eq. (5.19).

These decay photon fluxes provide again the ratios

$$\frac{\Phi(\hat{\nu}_{\rm e})}{\Phi(\hat{\nu}_\mu)}=\frac{\Omega_{\rm dm}}{\Omega_{\rm b}}=5.364, \qquad (12.12)$$

$$\frac{\Phi(\hat{\nu}_{\rm e})}{\Phi(\hat{\nu}_\tau)}=\frac{\Omega_{\rm dm}\,(1+z_{\rm MNA})}{\left\{\Omega_\Lambda/(1+z_{\rm MNA})^2\right\}-2\Omega_\nu\,(1+z_{\rm MNA})}=13.94\cong\frac{\Omega_\Lambda}{\Omega_{\rm b}} \qquad (12.13)$$

and

$$\frac{\Phi(\hat{\nu}_\mu)}{\Phi(\hat{\nu}_\tau)} = \frac{\Omega_b\,(1+z_{MNA})}{\left\{\Omega_\Lambda/(1+z_{MNA})^2\right\} - 2\,\Omega_\nu\,(1+z_{MNA})} = 2.598 \cong \frac{\Omega_\Lambda}{\Omega_{dm}}. \quad (12.14)$$

in excellent agreement with the predictions (9.14) to (9.16), respectively.

Consequently, the probable photon fluxes (12.9) to (12.11) of these 3 sterile neutrinos are coupled with the corresponding cosmological particle density parameters of the universe.

13 The relationship between big bang and final state of universe

In this chapter, we confirm the relationship between the big bang and the final state of the universe, introduced in Refs. [4-7]. For this goal, we assume that the corresponding present cosmological particle density parameters, which describe uniquely and precisely the past of the universe, define also the future of the universe via the Friedmann-Lemaitre Equations.

Thus, the decelerated exponential expansion (10.7) yields its acceleration to

$$\ddot{R} = \Omega_\Lambda H_0^2 R\,, \quad (13.1)$$

so that we have an effective equilibrium $\ddot{R} + \ddot{R}_{MNA} = 0$ between the mean negative acceleration $\ddot{R}_{MNA} = (-8.88 \pm 0.43)\cdot 10^{-8}\,\mathrm{cm\,s^{-2}}$ [see Eq. (11.6)] and the result (13.1) at the effective scale factor

$$R_{eff} = \frac{-\ddot{R}_{MNA}}{\Omega_\Lambda H_0^2} = (2.72 \pm 0.13)\cdot 10^{26}\,\mathrm{m}\,, \quad (13.2)$$

so that we have the blueshift condition

$$1 + z_{eff} = \frac{R_0}{R_{eff}} = 0.504 \pm 0.024\,. \quad (13.3)$$

Consequently, because of the result (10.8), the corresponding dark (d) energy E_d must be found via the dark energy density to

$$\rho_\Lambda c^2 = \Omega_\Lambda \rho_{0C} c^2 = (3.280 \pm 0.033)\cdot 10^3\,\mathrm{eV\,cm^{-3}}\,. \quad (13.4)$$

Therefore, according to Refs. [4-6], we apply the approach

$$E_d = \Omega_\Lambda \rho_{0C} c^2 \tfrac{4}{3}\pi d_{eff}^3\,, \quad (13.5)$$

where the angular-diameter distance d_{eff} is defined by

$$d_{\text{eff}} = R_{\text{eff}}\, r_{\text{eff}} = R_{\text{eff}}\, r \tag{13.6}$$

with $r = r_{\text{eff}}$ as the dimensionless time-independent comoving coordinate distance [9, 10].

As the probable origin of the dark energy, the 3 sterile neutrino types $\hat{\nu}_\tau$, $\hat{\nu}_\mu$ and $\hat{\nu}_e$ are assumed {see, e.g., Ref. [6] or the results (9.14) to (9.16) as well as (12.9) to (12.11)}. However, for the general considerations of the present chapter, we assume no special properties for these sterile neutrinos as one's own species according to Sec. 9. Thus, we assume that the dark energy E_d must be distributed among the decay products of these sterile neutrinos ($\hat{\nu}$).

Taking the decay condition of these sterile neutrino types via the gravitation into one photon with the energy of their half rest energy $\frac{1}{2} E_0(\hat{\nu})$ and one sterile neutrino relic with the energy of their half rest energy $\frac{1}{2} E_0(\hat{\nu})$ (see Refs. [2-6]), all these photons $kT_\gamma = \frac{1}{2} E_0(\hat{\nu})$ must define in average the half dark energy $\frac{1}{2} E_d$ as the greatest possible thermal photon energy kT according to

$$kT = \tfrac{1}{2} E_d = \sum_\gamma \left\langle kT_\gamma \right\rangle = \sum_\gamma \left\langle \tfrac{1}{2} E_0(\hat{\nu}) \right\rangle . \tag{13.7}$$

Therefore, applying the new thermal equilibrium [4-6] between the photons $kT_{\frac{1}{2}\hat{\nu}}$ and the sterile neutrino relics $\frac{1}{2} E_0(\hat{\nu})$, because of the energy conservation, all these sterile neutrino relics $\frac{1}{2} E_0(\hat{\nu}) = kT_{\frac{1}{2}\hat{\nu}}$ must also form in average the other half of the dark energy $\frac{1}{2} E_d$ as the greatest possible thermal energy kT according to

$$kT = \tfrac{1}{2} E_d = \sum_{\frac{1}{2}\hat{\nu}} \left\langle kT_{\frac{1}{2}\hat{\nu}} \right\rangle = \sum_{\frac{1}{2}\hat{\nu}} \left\langle \tfrac{1}{2} E_0(\hat{\nu}) \right\rangle . \tag{13.8}$$

since the corresponding energy $kT_{\frac{1}{2}\hat{\nu}}$ of the sterile neutrino relics must be equivalent to kT_γ of the corresponding photons.

The special nature of the decay product $\frac{1}{2} E_0(\hat{\nu}) = kT_{\frac{1}{2}\hat{\nu}}$ is explained by Eq. (3.135) of Ref. [6] as gravitons, assuming a decay of the sterile neutrino types as pairs with opposite spins in photons and gravitons (see also Ref. [7]).

Thus, the total gravitational potential energy $V_{\rm gr}(d_{\rm eff})$ of these decay products can be defined by

$$V_{\rm gr}(d_{\rm eff}) = \frac{G_N}{c^4} \frac{(\frac{1}{2} E_{\rm d})^2}{d_{\rm eff}} = \frac{\hbar c}{E_{\rm Pl}^2} \frac{(\frac{1}{2} E_{\rm d})^2}{d_{\rm eff}}. \tag{13.9}$$

For $V_{\rm gr}(d_{\rm eff}) = \frac{1}{2} E_{\rm d}$, we obtain

$$d_{\rm eff} = \frac{G_N}{c^4} \frac{1}{2} E_{\rm d} = \frac{\hbar c}{E_{\rm Pl}^2} \frac{1}{2} E_{\rm d}. \tag{13.10}$$

By Eqs. (13.5) and (13.10), we find the condition

$$d_{\rm eff} = \frac{\hbar c}{E_{\rm Pl}^2} \frac{1}{2} \Omega_\Lambda \rho_{0\rm C} c^2 \frac{4}{3} \pi d_{\rm eff}^3 \tag{13.11}$$

with the solution

$$d_{\rm eff} = \left(\frac{E_{\rm Pl}^2}{\hbar c \frac{1}{2} \Omega_\Lambda \rho_{0\rm C} c^2 \frac{4}{3} \pi} \right)^{\frac{1}{2}} = (3.316 \pm 0.017) \cdot 10^{26}\ \text{m}, \tag{13.12}$$

so that Eq. (13.6) gives the reasonable value

$$r = r_{\rm eff} = \frac{d_{\rm eff}}{R_{\rm eff}} = 1.2191, \tag{13.13}$$

i.e. the lifetime $\tau_{\hat{\nu}}$ of the sterile neutrinos [5, 6] is defined by

$$\tau_{\hat{\nu}} = \frac{d_{\rm eff}}{c} = (1.1061 \pm 0.0057) \cdot 10^{18}\ \text{s} = 35.05 \pm 0.18\ \text{Gyr}. \tag{13.14}$$

Using Eqs. (13.5) and (13.12), we get quantitatively the dark energy

$$E_{\rm d} = \Omega_\Lambda \rho_{0\rm C} c^2 \frac{4}{3} \pi d_{\rm eff}^3 = (5.010 \pm 0.031) \cdot 10^{89}\ \text{eV}, \tag{13.15}$$

so that the expressions (13.7) and (13.8) yield

$$kT = \frac{1}{2} E_{\rm d} = \sum_\gamma \langle kT_\gamma \rangle = (2.505 \pm 0.016) \cdot 10^{89}\ \text{eV} \tag{13.16}$$

and

$$kT = \tfrac{1}{2} E_{\mathrm{d}} = \sum_{\frac{1}{2}\hat{\nu}} \left\langle kT_{\frac{1}{2}\hat{\nu}} \right\rangle = (2.505 \pm 0.016) \cdot 10^{89}\ \mathrm{eV}\,. \tag{13.17}$$

Then, taking, for example, the general expressions (3.23) and (3.24) of Ref. [6] for the distance R_{BB} and the time t_{BB} of the big bang (BB), derived by aid of the quantum gravity [4, 5], we find

$$R_{\mathrm{BB}} = \left(\frac{\pi^4\, \Omega_\gamma}{292.2273} \right)^{1/3} \frac{\hbar c}{\frac{1}{2} E_{\mathrm{d}}} = (2.070 \pm 0.023) \cdot 10^{-98}\ \mathrm{m} \tag{13.18}$$

and

$$t_{\mathrm{BB}} = \frac{R_{\mathrm{BB}}}{c} = \left(\frac{\pi^4\, \Omega_\gamma}{292.2273} \right)^{1/3} \frac{\hbar}{\frac{1}{2} E_{\mathrm{d}}} = (6.905 \pm 0.077) \cdot 10^{-107}\ \mathrm{s}\,. \tag{13.19}$$

Consequently, the beginning (big bang) of the universe is determined by the energy uncertainty

$$\widetilde{E}_{\mathrm{BB}} = \hbar / 2 t_{\mathrm{BB}} = (4.766 \pm 0.053) \cdot 10^{90}\ \mathrm{eV} \tag{13.20}$$

in excellent agreement with the result (3.11) of Ref. [7].

Then, using, for example, the result (5.10) of Ref. [7], we get for the near massless rest energy of the photons and gravitons

$$E_0(\gamma) = E_0(\mathrm{G}) = \frac{E_{\mathrm{Pl}}^2}{\hbar c} R_{\mathrm{BB}} = (1.564 \pm 0.017) \cdot 10^{-35}\ \mathrm{eV} \tag{13.21}$$

in excellent agreement with the results (3.27) and (6.2) of Ref. [6].

According to the result (3.116) of Ref. [6], in dependence of the temperature, the quantum gravity yields also the vacuum energy density to

$$\widehat{\rho}_{\mathrm{vac}}(T)\, c^2 = \frac{2}{3} \Omega_\gamma \frac{\pi^2}{15} \frac{E_{\mathrm{Pl}}^6}{(\hbar c)^3\, (kT)^2}\,. \tag{13.22}$$

For the relationship between the big bang and final (f) state of the universe, because of the condition (3.120) of Ref. [6], we can set $kT = kT_{\mathrm{BB}} = \widetilde{E}_{\mathrm{BB}}$ [see Eq. (13.20)], so that via Eq. (13.22) we must obtain $\widehat{\rho}_{\mathrm{vac}}(T_{\mathrm{BB}}) = \rho_{\mathrm{f}}$, i.e. the energy density of this final state is found to

$$\rho_{\mathrm{f}}\, c^2 = \frac{2}{3} \Omega_\gamma \frac{\pi^2}{15} \frac{E_{\mathrm{Pl}}^6}{(\hbar c)^3\, \widetilde{E}_{\mathrm{BB}}^2} = (4.53 \pm 0.10) \cdot 10^{-4}\ \mathrm{eV\,cm^{-3}}\,. \tag{13.23}$$

Consequently, because of the energy conservation of E_{d}, we must assume

$$\rho_{\mathrm{f}} = \Omega_{\Lambda}\,\rho_{0\mathrm{C}}\left(\frac{d_{\mathrm{eff}}}{d_{\mathrm{f}}}\right)^{3}, \tag{13.24}$$

where d_{f} is assumed as the angular-diameter distance of the final state of the universe, so that d_{f} is determined by

$$d_{\mathrm{f}} = \left(\frac{\Omega_{\Lambda}\,\rho_{0\mathrm{C}}}{\rho_{\mathrm{f}}}\right)^{1/3} d_{\mathrm{eff}} = (6.415 \pm 0.033)\cdot 10^{28}\ \mathrm{m}. \tag{13.25}$$

Then, the age t_{f} of the final state of the universe is given by

$$t_{\mathrm{f}} = \frac{d_{\mathrm{f}}}{c} = (2.140 \pm 0.011)\cdot 10^{20}\ \mathrm{s} = 6781 \pm 35\ \mathrm{Gyr}. \tag{13.26}$$

Assuming the decay of 3 sterile neutrino pairs with opposite spin directions into two photons and two gravitons (see above), by the result (13.21), their Compton distance (wavelength) is given by

$$d_{\lambda\!\!\!^{-}} = \frac{\hbar c}{2\{3E_0(\gamma) + 3E_0(\mathrm{G})\}} = (1.052 \pm 0.011)\cdot 10^{27}\ \mathrm{m}. \tag{13.27}$$

Then, via the results (13.25) and (13.27), we can introduce the angular-diameter distance

$$d_{\mathrm{f}}^{*} = d_{\mathrm{f}} - d_{\lambda\!\!\!^{-}} = (6.315 \pm 0.044)\cdot 10^{28}\ \mathrm{m}, \tag{13.28}$$

so that because of Eqs. (13.23) and (13.24) we obtain the new energy density

$$\rho_{\mathrm{f}}^{*}c^{2} = \Omega_{\Lambda}\,\rho_{0\mathrm{C}}\,c^{2}\left(\frac{d_{\mathrm{eff}}}{d_{\mathrm{f}}^{*}}\right)^{3} = (4.75 \pm 0.38)\cdot 10^{-4}\ \mathrm{eV\,cm^{-3}}. \tag{13.29}$$

Thus, for the ratio $\rho_{\mathrm{f}}^{*}/\rho_{\mathrm{f}}$, we get

$$\frac{\rho_{\mathrm{f}}^{*}}{\rho_{\mathrm{f}}} = \left(\frac{d_{\mathrm{f}}}{d_{\mathrm{f}}^{*}}\right)^{3} = \frac{V_{\mathrm{f}}}{V_{\mathrm{f}}^{*}} = 1.05 \pm 0.11, \tag{13.30}$$

where $V_{\mathrm{f}} \equiv \tfrac{4}{3}\pi(d_{\mathrm{f}})^{3}$ and $V_{\mathrm{f}}^{*} \equiv \tfrac{4}{3}\pi(d_{\mathrm{f}}^{*})^{3}$.

According to the result (3.69) of Ref. [6], for the final state of the universe, a temperature ratio, derived by aid of two different values of the vacuum energy density of the universe, was found to

$$\frac{T_1}{T_2}=\left[\frac{3N(T)}{2}\right]^{1/4}\cong 1.017, \tag{13.31}$$

where $N(T)\equiv(4/11)^{1/3}$ (see Sec. 6). Consequently, in this work, for the states V_{f} and V_{f}^{*}, the entropy conservation of these temperatures agrees with

$$\left(\frac{T_1}{T_2}\right)^3=\frac{V_{\mathrm{f}}}{V_{\mathrm{f}}^{*}}\cong 1.05\pm 0.11. \tag{13.32}$$

With that, for the relationship between the big bang and the final state of the universe, we have shown that the most important results of the Refs. [4-6] are confirmed. Besides, we have proved that the relationship of the big bang and the final state of the universe is independent on the special properties of the probable 3 sterile neutrinos as one's own species.

Consequently, because of the result (13.21), for the results (13.16) and (13.17), it is not necessary to take into account the two absolute numbers $N_\gamma(\hat{\nu})=\frac{1}{2}E_{\mathrm{d}}/E_0(\gamma)=1.602\cdot 10^{124}$ and $N_{\mathrm{G}}(\hat{\nu})=\frac{1}{2}E_{\mathrm{d}}/E_0(\mathrm{G})=1.602\cdot 10^{124}$ for the decay products of the sterile neutrinos, which are characterized by their relativistic energies $\frac{1}{2}\sum_i E_0^{\gamma}(\hat{\nu}_i)=13459\,\mathrm{eV}$ and $\frac{1}{2}\sum_i E_0^{\mathrm{G}}(\hat{\nu}_i)=13459\,\mathrm{eV}$, i.e. $N_\gamma(\hat{\nu})$ and $N_{\mathrm{G}}(\hat{\nu})$ are also found via Eqs. (9.11) to (9.13) to

$$N_\gamma(\hat{\nu})=\frac{N_0^{\gamma}(\hat{\nu})\,\frac{1}{2}\sum_i E_0^{\gamma}(\hat{\nu}_i)}{E_0(\gamma)}=\frac{1}{2}E_{\mathrm{d}}/E_0(\gamma)=1.602\cdot 10^{124} \tag{13.33}$$

and

$$N_{\mathrm{G}}(\hat{\nu})=\frac{N_0^{\mathrm{G}}(\hat{\nu})\,\frac{1}{2}\sum_i E_0^{\mathrm{G}}(\hat{\nu}_i)}{E_0(\mathrm{G})}=\frac{1}{2}E_{\mathrm{d}}/E_0(\mathrm{G})=1.602\cdot 10^{124} \tag{13.34}$$

with

$$N_0^{\gamma}(\hat{\nu})=\frac{\frac{1}{2}E_{\mathrm{d}}}{\frac{1}{2}\sum_i E_0^{\gamma}(\hat{\nu}_i)}=1.861\cdot 10^{85} \tag{13.35}$$

and

$$N_0^{\mathrm{G}}(\hat{\nu})=\frac{\frac{1}{2}E_{\mathrm{d}}}{\frac{1}{2}\sum_i E_0^{\mathrm{G}}(\hat{\nu}_i)}=1.861\cdot 10^{85}, \tag{13.36}$$

since we must consider so only the two quantities $\frac{1}{2} E_{\mathrm{d}} = N_{\gamma}(\hat{\nu})\, E_0(\gamma)$ and $\frac{1}{2} E_{\mathrm{d}} = N_{\mathrm{G}}(\hat{\nu})\, E_0(\mathrm{G})$, respectively.

14 The direct estimation of the age of the final state of the universe via the sterile neutrinos

In this chapter, in contrast to the Refs. [5, 6], we estimate correctly the age of the final state of the universe via the decay of the sterile neutrinos as one's own species according to their present number density $n_0(\hat{\nu}) = 0.1776\,\mathrm{cm}^{-3}$ of Eqs. (9.4) to (9.6), which is confirmed excellently by a more physical principle according to the results (9.17) to (9.26).

For this goal, considering the Eqs. (13.12) and (13.16) and the results of the Refs. [4-6], we can form the thermal photon density $n_{\gamma}(d_{\mathrm{eff}})$ via

$$n_{\gamma}(d_{\mathrm{eff}}) = \frac{\pi^2}{15}\left(\frac{\frac{1}{2} E_{\mathrm{d}}}{\hbar c}\right)^3 = (1.346 \pm 0.025)\cdot 10^{282}\ \mathrm{cm}^{-3}\,. \qquad (14.1)$$

Because of the decay of one sterile neutrino type is only possible via a corresponding sterile neutrino pair with opposite spin directions into two photons and two gravitons [see Eq. (13.27)], we must assume as the correct sterile neutrino number density $n_{\hat{\nu}}(d_{\mathrm{eff}}) = 2\, n_{\gamma}(d_{\mathrm{eff}})$.

Because of the energy conservation, this sterile neutrino number density $n_{\hat{\nu}}(d_{\mathrm{eff}})$ must also agree with the sterile neutrino number density $n_{\hat{\nu}}(d_{\mathrm{f}})$ of the final state of the universe, i.e. we have

$$n_{\hat{\nu}}(d_{\mathrm{f}}) = n_{\hat{\nu}}(d_{\mathrm{eff}}) = 2\, n_{\gamma}(d_{\mathrm{eff}}) = (2.692 \pm 0.050)\cdot 10^{282}\ \mathrm{cm}^{-3}\,. \qquad (14.2)$$

Then, by the universal decay law, because of the decay of the 3 sterile neutrino type pairs with opposite spin directions [see Eq. 13.27)], the initial (i) decay photon and graviton number densities $n_{\gamma}(d_{\mathrm{i}})$ and $n_{\mathrm{G}}(d_{\mathrm{i}})$ are defined by

$$2\left\{n_{\gamma}(d_{\mathrm{i}}) + n_{\mathrm{G}}(d_{\mathrm{i}})\right\} V(\hat{\nu}) = n_{\hat{\nu}}(d_{\mathrm{f}})\, V(\hat{\nu})\, \mathrm{e}^{-t_{\mathrm{f}}/\tau_{\hat{\nu}}}\,. \qquad (14.3)$$

Now, we determine the unknown quantities $\left\{n_{\gamma}(d_{\mathrm{i}}) + n_{\mathrm{G}}(d_{\mathrm{i}})\right\}$ and $V(\hat{\nu})$.

Firstly, we estimate $n_\gamma(d_i)$ and $n_G(d_i)$. For this goal, we use the fact that the present day universe contains a relic neutrino (ν) background with the temperature $T_{\nu,0} = (4/11)^{1/3} T_0 = 1.945\,\mathrm{K}$ (see Refs. [5, 6, 9, 10]), where ν describes the light neutrinos, which were assumed as nearly massless.

For the formation of the present cosmological particle density parameters, which were identified as the cosmological parameters of the heavy neutrinos (see Sec. 8), the necessary energy (for the conversion of the light neutrinos into heavy neutrinos) was taken from the early universe, so that the light and the heavy neutrinos must possess the same number density, i.e. $n_0(\nu) = n_0(\tilde{\nu}) = 112\,\mathrm{cm}^{-3}$ [see Eqs. (6.1) as well as (8.10) to (8.12)].

Consequently, the light (ν) and the heavy ($\tilde{\nu}$) neutrino background must possess the same temperature $T_{\nu,0} = T_{\tilde{\nu},0} = 1.945\,\mathrm{K}$ according to the ideal gas law because of the constant pressure.

At the pressure P, between the energy E and the volume V, for the expansion of the massive universe, the kinetic theory of gases yields

$$dE = -P\,dV\,. \tag{14.4}$$

At the formation of sterile neutrinos, the necessary energy is taken from the relic neutrino ($\tilde{\nu}$) background, so that the number density of the probable sterile neutrinos ($\hat{\nu}$) decreases to $n_0(\hat{\nu}) = 0.1776\,\mathrm{cm}^{-3}$ [see Eqs. (9.4) to (9.6)].

Therefore, according to Eq. (14.4), at constant pressure, for the heavy neutrino relic of $T_{\tilde{\nu},0} = 1.945\,\mathrm{K}$, we must here assume the small volume changing proportional to $-1/n_0(\hat{\nu})$, so that we get a cooling of this relic neutrino background from $T_{\tilde{\nu},0} = 1.945\,\mathrm{K}$ to $T_{\hat{\nu},0}$ by the large volume changing proportional to $-1/n_0(\tilde{\nu})$, so that the ideal gas law yields

$$T_{\hat{\nu},0} = \frac{n_0(\hat{\nu})}{n_0(\tilde{\nu})} T_{\tilde{\nu},0} = (3.084 \pm 0.057) \cdot 10^{-3}\ \mathrm{K}\,, \tag{14.5}$$

i.e. the corresponding thermal photon energy is given by

$$kT = k\,(11/4)^{1/3} T_{\hat{\nu},0} = (3.723 \pm 0.069) \cdot 10^{-7}\ \mathrm{eV}\,. \tag{14.6}$$

Then, using Eqs. (2.40) or (3.32) of Ref. [6], for the sterile neutrinos, the quantum gravity [4, 5] yields the distance

$$R = R_{\hat{\nu}} = \frac{\hbar c}{E_{\mathrm{Pl}}^2} k\,(11/4)^{1/3} T_{\hat{\nu},0} = (4.928 \pm 0.091) \cdot 10^{-68}\ \mathrm{cm}\,, \tag{14.7}$$

so that Eqs. (2.44) or (3.33) of Ref [6] give correspondingly the number density $n_{\hat{\nu}}(R_{\hat{\nu}})$ of these sterile neutrinos, which must be identical with the thermal number density $n_{\gamma}(R_{\hat{\nu}})$ of the massless photons according to Eqs. (2.46) or (3.33) of Ref. [6], i.e. we have

$$n_{\hat{\nu}}(R_{\hat{\nu}}) = n_{\gamma}(R_{\hat{\nu}}) = \frac{2}{3}\Omega_{\gamma}\frac{\pi^2}{15}\frac{1}{R_{\hat{\nu}}^3} = (1.99 \pm 0.11) \cdot 10^{197}\ \mathrm{cm}^{-3}\,. \tag{14.8}$$

Consequently, for these 3 sterile neutrino types, the initial decay photon and graviton number densities $n_{\gamma}(d_{\mathrm{i}})$ and $n_{\mathrm{G}}(d_{\mathrm{i}})$ are determined, for example, by

$$\left\{n_{\gamma}(d_{\mathrm{i}}) + n_{\mathrm{G}}(d_{\mathrm{i}})\right\} = \left\{3+3\right\} n_{\hat{\nu}}(R_{\hat{\nu}}) = 6\, n_{\hat{\nu}}(R_{\hat{\nu}}) =$$

$$= 4\,\Omega_{\gamma}\frac{\pi^2}{15}\frac{1}{R_{\hat{\nu}}^3} = (1.197 \pm 0.067) \cdot 10^{198}\ \mathrm{cm}^{-3}\,. \tag{14.9}$$

Secondly, we determine the constant sterile neutrino volume $V(\hat{\nu})$. We assume that the sterile neutrinos of the dark energy $E_{\mathrm{d}} = 5.010 \cdot 10^{89}\,\mathrm{eV}$ of Eq. (13.15) occupy the volume V_{f} in the final state of the universe, i.e. we have

$$V_{\mathrm{f}} = \frac{E_{\mathrm{d}}}{\Omega_{\Lambda}\,\rho_{0\mathrm{C}}\,c^2} = (1.527 \pm 0.015) \cdot 10^{86}\ \mathrm{cm}^3\,. \tag{14.10}$$

Using the present number density $n_0(\hat{\nu}) = 0.1776 \pm 0.0033\,\mathrm{cm}^{-3}$ of these sterile neutrinos (see above), we can introduce their constant volume $V(\hat{\nu})$ to

$$V(\hat{\nu}) = \frac{\mathrm{const}(\hat{\nu})}{n_0(\hat{\nu})}\,, \tag{14.11}$$

so that for the final state of the universe the number N_{f} of the sterile neutrinos is given by

$$N_{\mathrm{f}} = \frac{V_{\mathrm{f}}}{V(\hat{\nu})} = \frac{n_0(\hat{\nu})\,V_{\mathrm{f}}}{\mathrm{const}(\hat{\nu})} = \frac{2.713 \cdot 10^{85}}{\mathrm{const}(\hat{\nu})}\,. \tag{14.12}$$

Because of the enormous age of the final state of the massive universe denoted as classical universe in Ref. [7], we can assume a complete decay of the N_f sterile neutrinos according to the expression (14.3).

Consequently, in the massless universe [6] denoted in Ref. [7] as quantum universe, the beginning of the sterile neutrino decay, which can only take place for sterile neutrino pairs with opposite spins, is then defined by 6 photons and 6 gravitons because of the 3 sterile neutrino types, so that instead of the volume V_f these 12 different decay products must yield the initial (i) volume V_i of the sterile neutrinos to

$$V_i = \frac{12}{n_0(\hat{\nu})} = 67.6 \pm 1.3\ \mathrm{cm}^3 . \tag{14.13}$$

Thus, the initial number $2N_i$ of the sterile neutrinos of the massless universe must be defined by

$$2N_i = \frac{2V_i}{V(\hat{\nu})} = \frac{24}{\mathrm{const}(\hat{\nu})} . \tag{14.14}$$

Instead of Eq. (14.3), we can assume

$$2N_i = N_f\, e^{-t_f/\tau_{\hat{\nu}}} , \tag{14.15}$$

so that because of Eqs. (14.2) as well as (14.9) to (14.15) we have

$$N_f = V(\hat{\nu})\, n_{\hat{\nu}}(d_f) = \frac{n_0(\hat{\nu})\, V_f}{\mathrm{const}(\hat{\nu})} = \frac{\mathrm{const}(\hat{\nu})}{n_0(\hat{\nu})} n_{\hat{\nu}}(d_f) \tag{14.16}$$

and

$$2N_i = V(\hat{\nu})\, 12\, n_{\hat{\nu}}(R_{\hat{\nu}}) = \frac{2V_i}{V(\hat{\nu})} = \frac{24}{\mathrm{const}(\hat{\nu})} = \frac{\mathrm{const}(\hat{\nu})}{n_0(\hat{\nu})} 12\, n_{\hat{\nu}}(R_{\hat{\nu}}) . \tag{14.17}$$

Then, the expressions (14.16) and (14.17) provide the two conditions

$$\frac{n_0(\hat{\nu})\, V_f}{\mathrm{const}(\hat{\nu})} = \frac{\mathrm{const}(\hat{\nu})}{n_0(\hat{\nu})} n_{\hat{\nu}}(d_f) \tag{14,18}$$

and

$$\frac{24}{\mathrm{const}(\hat{\nu})} = \frac{\mathrm{const}(\hat{\nu})}{n_0(\hat{\nu})} 12\, n_{\hat{\nu}}(R_{\hat{\nu}}) . \tag{14.19}$$

Thus, using the results (14.2) and (14.8) as well as (14.9) and (14.10), the conditions (14.18) or (14.19) yield

$$\mathrm{const}(\hat{\nu}) = \left(\frac{n_0^2(\hat{\nu})\, V_\mathrm{f}}{n_{\hat{\nu}}(d_\mathrm{f})} \right)^{1/2} = (1.338 \pm 0.042) \cdot 10^{-99} \tag{14.20}$$

or

$$\mathrm{const}(\hat{\nu}) = \left(\frac{2\, n_0(\hat{\nu})}{n_{\hat{\nu}}(R_{\hat{\nu}})} \right)^{1/2} = (1.336 \pm 0.049) \cdot 10^{-99}\,, \tag{14.21}$$

so that Eq. (14.11) leads to the constant sterile neutrino volume

$$V(\hat{\nu}) = \frac{\mathrm{const}(\hat{\nu})}{n_0(\hat{\nu})} = (7.53 \pm 0.39) \cdot 10^{-99}\ \mathrm{cm}^3\,. \tag{14.22}$$

Taking Eqs. (14.16) and (14.17) and using the results (14.2) and (14.9) as well as (14.22), we obtain the particle numbers

$$N_\mathrm{f} = V(\hat{\nu})\, n_{\hat{\nu}}(d_\mathrm{f}) \cong (2.03 \pm 0.11) \cdot 10^{184} \tag{14.23}$$

and

$$2\, N_\mathrm{i} = V(\hat{\nu})\, 12\, n_{\hat{\nu}}(R_{\hat{\nu}}) \cong (1.80 \pm 0.10) \cdot 10^{100}\,. \tag{14.24}$$

Consequently, by Eqs. (14.2) and (14.9), considering the expressions (14.10) as well as (14.23) and (14.24), the results (14.25) and (14.26) yield the following connections for the constant volume $V(\hat{\nu})$ of the sterile neutrinos

$$V(\hat{\nu}) = \frac{V_\mathrm{f}}{N_\mathrm{f}} = \frac{N_\mathrm{f}}{n_{\hat{\nu}}(d_\mathrm{f})} \cong 7.53 \cdot 10^{-99}\ \mathrm{cm}^3 \tag{14.25}$$

or

$$V(\hat{\nu}) = \frac{V_\mathrm{i}}{N_\mathrm{i}} = \frac{N_\mathrm{i}}{6\, n_{\hat{\nu}}(R_{\hat{\nu}})} \cong 7.53 \cdot 10^{-99}\ \mathrm{cm}^3 \tag{14.26}$$

in accordance with Eq. (14.22).

Thus, by the assumptions (14.3) and (14.15), the expressions (14.23) and (14.24) confirm the estimated age $t_\mathrm{f} = (2.140 \pm 0.011) \cdot 10^{20}\ \mathrm{s} = 6781 \pm 35\ \mathrm{Gyr}$ of Eq. (13.26) for the final state of the universe, since we have also

$$t_\mathrm{f} = \tau_{\hat{\nu}} \ln \frac{N_\mathrm{f}}{2N_\mathrm{i}} = \tau_{\hat{\nu}} \ln \frac{n_{\hat{\nu}}(d_\mathrm{f})}{12\, n_{\hat{\nu}}(R_{\hat{\nu}})} = (2.140 \pm 0.019) \cdot 10^{20}\ \mathrm{s} \tag{14.27}$$

via the result (13.14), so that the angular-diameter distance of the final state of the universe is given again by

$$d_{\mathrm{f}} = c\,t_{\mathrm{f}} = (6.415 \pm 0.057) \cdot 10^{28}\ \mathrm{m} \tag{14.28}$$

in excellent agreement with the result (13.25).

With that, we have confirmed excellently the results of Sec. 13. Therefore, the excellent agreement of the results (13.26) and (14.27) supports the adopted and completed interpretation of the observations of Bulbul et al. (see chapters 9 and 12), since we have applied alone the present neutrino number density $n_0(\hat{\nu}) = 0.1776\,\mathrm{cm}^{-3}$ of Eqs. (9.4) to (9.6) for the 3 probable sterile neutrinos (as one's own species) in Eqs. (14.5) as well as (14.11) to (14.13).

15 The correctness of the new approximation of the estimation of the 3 light neutrino rest energies

In this chapter, we show the correctness of the new approximation of the estimation of the 3 light neutrinos. For this goal, we use our hypothesis [3-5] that the blueshift condition of the light electron neutrino (as smallest particle) determines the scale factor of the final state of the universe.

For the old and new approximation, the corresponding density parameters were found to $\Omega_\nu(\nu_{\mathrm{e}}) = (1.641 \pm 0.054) \cdot 10^{-5}\, h^{-2} = (3.61 \pm 0.17) \cdot 10^{-5}$ [see Eq. (7.6)] and $\Omega_\nu(\nu_{\mathrm{e}}) = (8.02 \pm 0.21) \cdot 10^{-7}\, h^{-2} = (1.765 \pm 0.072) \cdot 10^{-6}$ [see Eq. (7.10)], respectively. Thus, according to Refs. [2-6], for $N(T) \equiv 3.362244$ and $\Omega_\gamma = (5.44 \pm 0.08) \cdot 10^{-5}$ (see Table I), we can estimate the corresponding blueshift conditions to

$$1 + z_{\mathrm{old}} = \frac{2\,\Omega_\nu(\nu_{\mathrm{e}})}{N(T)\,\Omega_\gamma} = 0.395 \pm 0.013 \tag{15.1}$$

and

$$1 + z_{\mathrm{new}} = \frac{2\,\Omega_\nu(\nu_{\mathrm{e}})}{N(T)\,\Omega_\gamma} = 0.01929 \pm 0.00052, \tag{15.2}$$

so that the respective scale factors are given by

$$R_{\mathrm{old}} = \frac{R_0}{1 + z_{\mathrm{old}}} = (3.47 \pm 0.11) \cdot 10^{26}\ \mathrm{m} \tag{15.3}$$

and

$$R_{\text{new}} = \frac{R_0}{1+z_{\text{new}}} = (7.11 \pm 0.19) \cdot 10^{27}\ \text{m}. \tag{15.4}$$

Using the angular-diameter distance $d_{\text{f}} = (6.415 \pm 0.033) \cdot 10^{28}\ \text{m}$ [see Eq. (13.25)] of the final state of the universe, the corresponding dimensionless time-independent comoving coordinate distances r can be determined by

$$r = r_{\text{old}} = \frac{d_{\text{f}}}{R_{\text{old}}} = 184.9 \pm 6.8 \tag{15.5}$$

and

$$r = r_{\text{new}} = \frac{d_{\text{f}}}{R_{\text{new}}} = 8.65 \pm 0.28. \tag{15.6}$$

Considering the dimensionless time-independent comoving coordinate distances $r = r_{\text{eff}} = d_{\text{eff}}/R_{\text{eff}} = 1.2191$ [see Eq. (13.13)] of the effective scale factor (13.2), the value (15.5) is very high as compared with the result (15.6), so that the outcome (15.6) must be reasonable, i.e. the result (15.4) defines well the scale factor of the final state of the universe for $1+z_{\text{acc}} = 0.01929$ in accordance with Eq. (10.38).

For the correctness of these assumptions, a strong argument is the fact that the value $H_{\text{acc}} = H_{\text{old}} = 41.52\ \text{km s}^{-1}\text{Mpc}^{-1}$ at $1+z_{\text{acc}} = 1+z_{\text{old}} = 0.395$ does not reproduce the limiting value $\frac{1}{2}H_0$ by the result (10.27) in contrast to $H_{\text{acc}} = H_{\text{new}} = H_{\text{min}} = 33.75\ \text{km s}^{-1}\text{Mpc}^{-1}$ at $1+z_{\text{acc}} = 1+z_{\text{new}} = 0.01929$.

Thus, for the estimation of the 3 light neutrino rest energies, the new approximation must be realistic in contrast to the old approximation. This fact is in accordance with the SUSY GUT coupling constants (5.19) as well as (9.17) and (9.24). All these results confirm excellently the new model of the complex neutrino problem in the particle physics and the cosmology, derived in this work.

In this connection, the assumption of a slow linear expansion [4-6] is incorrect because it describes only approximately the final state of the universe by the unrealistic old approximation.

16 Summary

Because of improved experimental data, we have performed once more a thorough analysis of the actual complex (light, heavy and sterile) neutrino problem of the particle physics and the universe in connection with the present cosmological particle density parameters, whereat we have applied several results of our early works [1-7].

We have used a known old and a completely new approximation for the estimation of the 3 light neutrino rest energies. The old approximation is based on the known fictitious approach $E_0(\nu_\mu) = [E_0(\nu_\tau) \times E_0(\nu_e)]^{1/2}$. The new approximation is defined by an extreme value problem of the sum of the 3 light neutrino rest energies.

The old approximation yields $\sum_i E_0(\nu_i) \cong (6.066 \pm 0.052) \cdot 10^{-2}\,\mathrm{eV}$. Its separate neutrino rest energies are given by $E_0(\nu_e) \cong (1.544 \pm 0.049) \cdot 10^{-3}\,\mathrm{eV}$, $E_0(\nu_\mu) \cong (8.81 \pm 0.11) \cdot 10^{-3}\,\mathrm{eV}$ and $E_0(\nu_\tau) \cong (5.031 \pm 0.036) \cdot 10^{-2}\,\mathrm{eV}$. Their sum $\sum_i E_0(\nu_i) \cong (6.066 \pm 0.052) \cdot 10^{-2}\,\mathrm{eV}$ lies near to the lower boundary in the expected energy range $0.12\,\mathrm{eV} > \sum_i E_0(\nu_i) > 0.06\,\mathrm{eV}$

The new approximation gives $\sum_i E_0(\nu_i) \cong (5.904 \pm 0.046) \cdot 10^{-2}\,\mathrm{eV}$. Its separate neutrino rest energies are defined by $E_0(\nu_e) \cong (5.74 \pm 0.19) \cdot 10^{-5}\,\mathrm{eV}$, $E_0(\nu_\mu) \cong (8.68 \pm 0.10) \cdot 10^{-3}\,\mathrm{eV}$ and $E_0(\nu_\tau) \cong (5.028 \pm 0.036) \cdot 10^{-2}\,\mathrm{eV}$. Their sum $\sum_i E_0(\nu_i) \cong (5.904 \pm 0.046) \cdot 10^{-2}\,\mathrm{eV}$ does not lie in the expected energy range $0.12\,\mathrm{eV} > \sum_i E_0(\nu_i) > 0.06\,\mathrm{eV}$. However, it lies near to its lower limit.

Then, for the total present light neutrino density parameter of the universe, we have found $\Omega_\nu = (6.448 \pm 0.063) \cdot 10^{-4} h^{-2} = (1.419 \pm 0.035) \cdot 10^{-3}$ for the old approximation and $\Omega_\nu = (6.276 \pm 0.057) \cdot 10^{-4} h^{-2} = (1.382 \pm 0.033) \cdot 10^{-3}$ for the new approximation. These values lie also near to the lower boundary of the expected range $0.003 > \Omega_\nu > 0.0012$. Therefore, these facts are not critical, since the SUSY GUT coupling constant of the old approximation is

given by $\alpha_{GUT} = 0.03847 \pm 0.00049$, whereas the new approximation determines its value to $\alpha_{GUT} = 0.03952 \pm 0.00047$. Consequently, in contrast to the old approximation, the new approximation gives a better agreement with the well-known SUSY GUT coupling constant $\alpha_{GUT} \cong 0.04$, i.e. the new approximation must be more realistic.

The 3 independent present cosmological particle density parameters (Ω_b, Ω_{dm} and Ω_Λ), divided by the constant $A = (1.0630 \pm 0.0013) \cdot 10^{-2} h^{-2}\,\mathrm{eV}^{-1}$, yield the 3 rest energies of the heavy neutrinos to $E_0(\tilde{\nu}_e) \cong 2.104 \pm 0.017\,\mathrm{eV}$, $E_0(\tilde{\nu}_\mu) \cong 11.29 \pm 0.13\,\mathrm{eV}$ and $E_0(\tilde{\nu}_\tau) \cong 29.29 \pm 0.33\,\mathrm{eV}$, respectively. Their present number densities $n_0(\tilde{\nu}_i) = 112\,\mathrm{cm}^{-3}$ are identical with the present number densities $n_0(\nu_i) = 112.016 \pm 0.074\,\mathrm{cm}^{-3}$ of the light neutrinos.

By aid of the properties of these heavy neutrinos, using the observed data of several researcher groups for the photon decay of an unidentified X-ray emission line, attributed to the sterile neutrinos as dark matter candidates, we have also estimated the rest energies of the probable 3 sterile neutrinos $\hat{\nu}_i$ via the 3 independent present cosmological particle density parameters (Ω_b, Ω_{dm} and Ω_Λ), divided by the constant $B = 1.6854 \cdot 10^{-5} h^{-2}\,\mathrm{eV}^{-1}$, so that we have found their corresponding rest energies to $E_0(\hat{\nu}_e) \cong 1327\,\mathrm{eV}$, $E_0(\hat{\nu}_\mu) \cong 7120\,\mathrm{eV}$ and $E_0(\hat{\nu}_\tau) \cong 18470\,\mathrm{eV}$. Their present sterile neutrino number densities were determined to $n_0(\hat{\nu}_i) = 0.1776 \pm 0.0033\,\mathrm{cm}^{-3}$.

For the "present" exponential expansion, we have proved that the estimated Hubble constants H_{acc} decrease uniquely from a maximum value $H_{max} = 124.33\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, given for $1 + z_{acc} = 1.633$, to a minimum value $H_{min} = 33.75\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$, defined by $1 + z_{acc} = 0.01929$, whereat the value of the Hubble constant $H_0 = 67.4 \pm 0.5\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$ lies approximately in the middle of this range. Consequently, the complete range of these estimated Hubble constants defines a "present" decelerated exponential expansion of the universe, i.e. it is not accelerated.

The redshift condition $1+z_{\text{fix}}=1.085$, which is fixed by a delayed beginning of the "present" decelerated exponential expansion, yields the new present Hubble constant $H_{\text{acc}}=73.30\pm0.54\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$. This value agrees again excellently with the experimental values of the present Hubble constant, measured by Riess et al., whereat this value $H_{\text{acc}}=73.30\pm0.54\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$ represents their mean Hubble constant of $H=73.24\pm1.74\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$.

We have proved that the value $H_0=67.4\pm0.5\,\text{km}\,\text{s}^{-1}\,\text{Mpc}^{-1}$ of Planck 2013 defines precisely and uniquely the present Hubble constant of the present universe, so that all hitherto existing considerations can be conserved for the universe via the Friedmann-Lemaitre Equations, i.e. H_0 must be responsible further as its present-day Hubble expansion rate.

In excellent agreement with the observations, the successful evaluation of the mean negative acceleration (Pioneer anomaly), the astronomical unit changing and the flyby anomaly supports the estimated values of the Hubble constants. This connection confirms again the existence of this mean negative acceleration (Pioneer anomaly) as a real physical effect.

We have determined the end of the "present" decelerated exponential expansion as the final state of the universe with the age of $t_{\text{f}}=6781\,\text{Gyr}$ only via the decay of the sterile neutrinos, whereat we have shown that the relationship between the big bang and final state of the universe is independent on the special properties of the probable 3 sterile neutrinos as one's own species. However, the calculation of the same age $t_{\text{f}}=6781\,\text{Gyr}$ of the final state of the universe, found via the decay of the probable 3 sterile neutrinos as one's own species, supports the interpretation of the observations of Bulbul et al., since in this case we have used their number density $n_0(\hat{\nu})=0.1776\pm0.0033\,\text{cm}^{-3}$. Therefore, a new improved SUSY GUT coupling constant could be derived to $\alpha_{\text{GUT}}\rightarrow\alpha_{\text{GUT}}(\hat{\nu}_i,\tilde{\nu}_i)=0.03982\pm0.00037$ via $\alpha_{\text{GUT}}=E_0(\text{X})/E_0(\text{M})$, using $E_0(\text{X})\rightarrow E_0^{\hat{\nu}_i,\tilde{\nu}_i}(\text{X})=(2.725\pm0.025)\cdot10^{16}\,\text{GeV}$ and $E_0(\text{M})=(6.844\pm0.028)\cdot10^{17}\,\text{GeV}$ as the rest energy of the X gauge bosons and magnetic monopoles.

Via the angular-diameter distance $d_{\mathrm{f}} = ct_{\mathrm{f}} = 6.415 \cdot 10^{28}\,\mathrm{m}$ of the final state of the universe, using the scale factor of the new neutrino approximation, determined by the blueshift of its light electron neutrinos, we have proved that the new approximation of the estimation of the 3 light neutrino rest energies is more realistic in accordance with the coupling constants $\alpha_{\mathrm{GUT}} = 0.03952(47)$ and $\alpha_{\mathrm{GUT}} \to \alpha_{\mathrm{GUT}}(\hat{\nu}_i, \tilde{\nu}_i) = 0.03982(37)$, whereat the last value shows an excellent agreement with the SUSY coupling constant of $\alpha_{\mathrm{GUT}} \cong 0.04$, well-known by the extrapolation to the unification of the 3 low energy gauge couplings of the strong and electroweak interaction at the corresponding very high energies of about $3 \cdot 10^{16}\,\mathrm{GeV}$, i.e. $E_0(\mathrm{X}) \to E_0^{\hat{\nu}_i, \tilde{\nu}_i}(\mathrm{X}) = 2.725(25) \cdot 10^{16}\,\mathrm{GeV}$ is thus justified. These results are confirmed again by the final solution of $\alpha_{\mathrm{GUT}} \to \alpha_{\mathrm{GUT}}^{\mathrm{tot}} = 0.03983(47)$ and $n_0(\hat{\nu}_i) = \alpha_{\mathrm{GUT}}^2\, n_0(\tilde{\nu}_i) \to n_0^{\mathrm{tot}}(\hat{\nu}_i) = (\alpha_{\mathrm{GUT}}^{\mathrm{tot}})^2\, n_0(\tilde{\nu}_i) = 0.1777(42)\,\mathrm{cm}^{-3}$, which is independent on the "sterile neutrino" observations.

17 Acknowledgment

We acknowledge P. Ressel, who supported this work.

References

[1] H. Strusny, *The significance of the neutrinos and the magnetic monopoles for the universe*, Hadronic Journal **40**, 13 (2017)

[2] H. Strusny, *From particle-defined cosmological parameters or heavy neutrinos to the "joint" origin of dark matter and dark energy by sterile neutrino types*, Hadronic Journal **40**, 255 (2017)

[3] H. Strusny, *The role of light neutrinos and SUSY GUT particles as well as heavy and sterile neutrinos in the universe*, Hadronic Journal **40**, 375 (2017)

[4] H. Strusny, *Universe (particles) and anti-universe (antiparticles) as well as vacuum energy density or cosmological "constant" including quantum gravity*, Hadronic Journal **41**, 95 (2018)

[5] H. Strusny, *Big bang (by sterile neutrino decay) and quantum gravity in universe and corresponding anti-universe*, Hadronic Journal **42**, 383 (2019)

[6] H. Strusny, *Universe and time-reversal anti-universe as eternal cycle of evolution with photon rest energy, Hubble "constants" and time dependent cosmological "constant"*, Hadronic Journal **43**, 363 (2020)

[7] H. Strusny, *Quantum and classical universe and anti-universe*, Hadronic Journal **44**, 109 (2021)

[8] M. Treichel, *Teilchenphysik und Kosmologie* (Springer-Verlag, Berlin, Heidelberg, 2000)

[9] S. Weinberg, *Cosmology* (Oxford University Press, New York, 2008).

[10] H. Mo, F. van den Bosch, and S. White, *GALAXY FORMATION AND EVOLUTION*, (Cambridge University Press, Cambridge, UK, 2010).

[11] J. D. Anderson, P. A. Laing, E. L. Lau, A. S. Liu, M. M. Nieto, and S. G. Turyshev, *Phys. Rev.* D**65**, 082004 (2002)

[12] H. Dittus and C. Lämmerzahl, *Physik Journal* **5**, 25 (2006)

[13] G. A. Krasinsky and V. A. Brumberg, *Celest. Mech. & Dyn. Astron.* **90**, 267 (2004)

[14] E. M. Standish, *The astronomical unit now* in D. W. Kurtz (ed.): *Transits of Venus: New Views of the Solar System and Galaxy*, Proceedings IAU Colloquium No. 196, 2004 (Cambridge University Press, Cambridge, UK, 2005), p. 163-179

[15] J. D. Anderson and M. M. Nieto, *Astrometric solar-system anomalies*, in S. A. Klioner, P. K. Seidelman and M. H. Soffel (eds.): *Relativity in Fundamental Astronomy*, Proceedings IAU Symposium No. 261, 2009 (Cambridge University Press, Cambridge, UK, 2010), p. 189-197

[16] P. A. R. Ade et al., (Planck Collaboration I. and XVI.):
Planck 2013 results. I. Overview of products and scientific results, *Astron. & Astrophys.* **571**, A1 (2014), [http://arXiv.1303.5062v1];
Planck 2013 results. XVI. Cosmological parameters, *Astron. & Astrophys.* **571**, A16 (2014) [http://arXiv.1303.5076v1]

[17] J. Beringer *et al.* (Particle Data Group), *Phys.Rev.* D**86**, 010001 (2012), [http://pdg.lbl.gov]

[18] K. Olive *et al.* [Particle Data Group], *Chin. Phys.* C**38**, 090001 (2014), [http://pdg.lbl.gov].

[19] M. Tanabashi et al. [Particle Data Group], Phys. Rev. D**98**, 030001 (2018) and 2019 update [http://pdg.lbl.gov]

[20] P. A. Zyla et al. [Particle Data Group], Theor. Exp. Phys. **2020**, 083C01 (2020), [http://pdg.lbl.gov]

[21] K. Fujikawa and R. E. Shrock, *Phys. Rev. Lett.* B**45**, 963 (1980)

[22] A. Vilenkin and E. P. S. Shellard, *Cosmic Strings and other Topological Defects*, (Cambridge University Press, Cambridge, UK, 1994)

[23] C. Berger, *Elementarteilchenphysik*, (Springer-Verlag, Berlin, Heidedelberg, 2002)

[24] J. Ellis, S. Kelley and D. V. Nanopoulos, *Phys. Lett.* B**260**, 131 (1991)

[25] P. Langacker and M. Luo, *Phys. Rev.* D**44**, 817 (1991)

[26] P. Langacker and N. Polansky, *Phys. Rev.* D**47**, 4028 (1993)

[27] T. Blazek, M. Carena, S. Raby and C. E. M. Wagner, *Phys. Rev.* D**56**, 6919 (1997)

[28] E. Bulbul, M. Markevitch, A. Foster, R. K. Smith, M. Loewenstein and S. W. Randall, ApJ **789**, 13 (2014), [arXiv:1402.2301v2]

[29] A. Boyarsky, O. Ruchayskiy, D. Iakubovskyi, and J. Franse Phys. Rev. Lett. **113**, 251301 (2014), [arXiv:1402.4119]

[30] T. E. Jeltema and S. Profumo, Mon. Not. R. Astron. Soc. **450**, 2143 (2015), [arXiv:1408.1699]

[31] J. M. Cline and A. R. Frey, Phys. Rev. D**90**, 123537 (2014), [arXiv:1410.7766]

[32] S. K. Kang and A. Patra, arXiv:1412.4899

[33] D. A. Iakubovskyi, Adv. Astron. Space Phys. **6**, 3 (2016)

[34] A. G. Riess et al., arXiv:1604.01424

[35] J. D. Anderson, J. K. Campbell, J. E. Ekelund, J. Ellis, and J. F. Jordan, *Phys. Rev. Lett.* **100**, 091102 (2008)

[36] R. Rievers and C. Lämmerzahl, arXiv:1104.3985v2 [gr-qc], 5 May 2011

HADRONIC JOURNAL 45, 59 - 79 (2022)
DOI 10.29083/HJ.45.01.2022

INVESTIGATE THE EFFECT OF TEMPERATURE ON QUASIPARTICLE EXPECTATION VALUE AND PAIR CORRELATION FUNCTION FOR SINGLET WAVE IN AN OPEN QUANTUM DISSPATIVE SYSTEM AT NON-EQUILIBRIUM

Nabin Kumar Teemilsina[1,2], Saddam Husain Dhobi[1,2,3*], Kishori Yadav[1,2], Bibek Koirala[1,2], Padam Prakash Ghimire[1,2], Dilip Rokka[1,2], Suresh Prasad Gupta[1,2], Jeevan Jyoti Nakarmi[1,2]
[1]Department of Physics, Patan Multiple Campus, Tribhuvan University, Lalitpur-44700, Nepal
[2]Innovative Ghar Nepal, Lalitpur-44700, Nepal
[3]Robotics Academy of Nepal, Lalitpur-44700, Nepal
*saddam@ran.edu.np

Received March 3, 2022

Abstract

The objective of this work is to study the expectation value of quasiparticle and pair correlation function for quasiparticles for non-equilibrium. The expectation value for creation is found greater than annihilation at zero and non-zero temperature with quasiparticle excitation. For s-wave, d-wave and g-wave the expectation of quasiparticle for creation found greater than annihilation. The expectation value beyond 5K is constant and then decrease with increasing temperature, the maximum expectation value for creation and annihilation in region 1K to 5K. The expectation value for d-wave and g-wave found maximum at different angle. The maximum pair correlation function for quasiparticles is observed as 0.048 at $\varphi = 0$ for d-wave but at $\varphi = \frac{\pi}{2}$ the pair correlation function of d-wave and g-wave is same and at $\varphi = \frac{\pi}{3}, \frac{\pi}{6}$ the pair correlation function of d-wave is absence.

Keywords: Quasiparticles, expectation value, s-wave, d-wave, g-wave, pair correlation function.

Copyright © 2022 by Hadronic Press, Inc., Palm Harbor, FL 34684, USA

1. Introduction

Superconductivity can be regarded as super fluidity of electron gas. Super fluidity of helium and superconductivity in metals exist at very low temperatures. At this temperature, all motion of particles freeze, except zero-point oscillations. BCS theory of Superconductivity led to a better appreciation of the connection between super fluidity and superconductivity. Superfluidity of liquid ^{3}He originates from pairing of two ^{3}He atoms as in the case of Cooper's state in superconductivity. In an ordinary diatomic molecule, the state is spin singlet state if the total angular momentum quantum number l is even and the total spin $s = 0$. It is spin triplet if l is odd and $s = 1$ of p-wave. Normal metal and superconductor and normal liquid and superfluid are separated by a thermodynamic phase transitions. Each such phase transition can be characterized by the nature of the singularities in specific heat and other thermodynamic variables at the transition temperature T_c [1, 2].

Non-equilibrium physics exhibited by many-body systems, where the constituent particles behave quantum mechanically and strongly interact with one another. Modem technological advances in mesoscopic and atomic systems extend the study of non-equilibrium phenomena to the quantum domain. Dynamics in non-equilibrium phenomena is the current effort to develop quantum computers. The dynamics of critical phenomena are often best studied using both linear and nonlinear response to applied external probe fields. Non-equilibrium superconductivity has been an intriguing field of the modern solid state physics since the 1960s. The non-equilibrium state can be created by external disturbances such as laser irradiation, microwave irradiation, phonon injection, and quasiparticle injection by means of tunnel junctions, time-dependent externally applied currents and thermal gradients. The non-equilibrium state is characterized by a (non-equilibrium) distribution function, which is particle-hole symmetric; i.e. the populations of the electron-like and the hole-like branches of the quasiparticle spectrum are the same. If an electron-like ($k > k_F$) or a hole-like ($k < k_F$) excitation is added to a superconductor, a charge imbalance is created [3, 4].

Open quantum system is a quantum-mechanical system that interacts with an external quantum system, which is known as the environment or a bath. In general, these interactions significantly change the dynamics of the system and result in quantum dissipation, such that the information contained in the system is lost to its environment. Because no quantum system is completely isolated from its surroundings, it is important to develop a theoretical framework for treating these interactions in order to obtain an accurate understanding of quantum systems. Techniques developed in the context of open quantum systems have applicable in fields such as quantum optics,

quantum measurement theory, quantum statistical mechanics, quantum information science, quantum thermodynamics, quantum cosmology, quantum biology, and semi-classical approximations. It allows numerous applications which range over many branches of modern physics, from fundamental question of quantum mechanics to technological applications in quantum information processing. Some general applications area are Decaying Systems, Decoherence, Dissipation, and Noise. The density matrix is formally defined as the outer product of the wave function and it's conjugate.

$$\rho = |\psi(t)\rangle\langle\psi(t)| \tag{1}$$

This implies that, specify a state $|\chi\rangle$, the integral $\langle\chi|\rho|\chi\rangle$ gives the probability of finding a particle in the state $|\chi\rangle$. The statistical description of a classical observable obtained from moments of a probability distribution P, then ρ plays the role of P in the quantum case:

$$\langle \mathrm{A} \rangle = \int A\, P(A) dA = \langle\psi|\mathrm{A}|\psi\rangle = \mathrm{Tr}[\mathrm{A}\rho] \tag{2}$$

Where Tr [...] refers to tracing over the diagonal elements of the matrix. For a system described by a wave function

$$|\psi\rangle = \sum_n c_n(t)|n\rangle \tag{3}$$

The expectation value of an operator is

$$\langle\hat{A}(t)\rangle = \sum_{n,m} c_n(t)c_m^*(t)\langle m|\hat{A}|n\rangle \tag{4}$$

From equation (1) we obtain the elements of the density matrix as

$$\rho(t) = \sum_{n,m} c_n(t)c_m^*(t)|n\rangle\langle m| \equiv \sum_{n,m} \rho_{n,m}(t)\,|n\rangle\langle m| \tag{5}$$

We see that $\rho_{n,m}$, the density matrix elements, are made up of the time-evolving expansion coefficients. A pure state in a quantum system can be represented by a vector $|\psi\rangle$ in a complex vector space. The time evolution of the quantum state is described by the Schrodinger equation:

$$i\hbar\frac{d}{dt}|\psi\rangle = H|\psi\rangle \tag{6}$$

$$\frac{d}{dt}|\psi\rangle = \frac{-i}{\hbar}H|\psi\rangle \tag{7}$$

Where the Hamiltonian H is a Hermitian operator. The time evolution preserves the norm $|\psi\rangle\langle\psi|$ of a state. The equation of motion of density matrix follows naturally from the definition of ρ and the time- dependent Schrodinger equation.

$$\frac{\partial\rho}{\partial t} = \frac{\partial}{\partial t}[|\psi\rangle\langle\psi|] = \left[\frac{\partial}{\partial t}|\psi\rangle\right]\langle\psi| + |\psi\rangle\left[\frac{\partial}{\partial t}\langle\psi|\right] \quad (8)$$

From equation (7),

$$\frac{\partial}{\partial t}|\psi\rangle = \frac{-i}{\hbar}H|\psi\rangle \quad (9)$$

And

$$\frac{\partial}{\partial t}\langle\psi| = \frac{i}{\hbar}\langle\psi|H \quad (10)$$

Using these relation in equation (8),

$$\frac{\partial\rho}{\partial t} = \frac{-i}{\hbar}H|\psi\rangle\langle\psi| + \frac{i}{\hbar}|\psi\rangle\langle\psi|H = \frac{-i}{\hbar}[H\rho - \rho H] = \frac{-i}{\hbar}[H,\rho] \quad (11)$$

where $\rho = |\psi\rangle\langle\psi|$, equation (11) is also called the Liouville-Von Neuman equation.

The pure state of a quantum system is described by a wave function obeying the linear Schrodinger evolution equation or by a vector in the Hilbert space of states. For the description of mixed states, the notion of density matrix was introduced by von Neumann. A mixed state refers to any case in which subdivide a microscopic or macroscopic system into an ensemble, for which there is initially no phase relationship between the elements of the mixture. It include an ensemble at thermal equilibrium, and independently prepared states. The density matrix obeys the linear von Neumann evolution equation or quantum Liouville equation [5, 6].

2. Literature Review

BCS theory or Bardeen–Cooper–Schrieffer theory is the first microscopic theory of superconductivity since Heike Kamerlingh Onnes's 1911 discovery. The theory describes superconductivity as a microscopic effect caused by a condensation of Cooper pairs. The theory is also used in nuclear physics to describe the pairing interaction between nucleons in an atomic nucleus. A Cooper pair is a weak electron-electron bound pair mediated by a phonon interaction. Imagine an electron moving within a material. The Coulomb

attraction between the electron and the positively charged cores of ions in the material will leave a net positive charge in the vicinity. Pairs of electrons can behave very differently from single electrons which are fermions and must obey the Pauli Exclusion Principle. The pairs of electrons act more like bosons which can condense into the same energy level. The electron pairs have a slightly lower energy and leave an energy gap above them on the order of 0.001 eV which inhibits the kind of collision interactions which lead to ordinary resistivity. For temperatures such that the thermal energy is less than the band gap, the material exhibits zero resistivity.

Mean-field approximation is powerful analytical method to make a rough prediction of the macroscopic behavior of a complex system to analyze the collective behavior of a dynamical system comprising many interacting particles. It is widely used to study various kinds of large-scale dynamical systems. For a system having large number of dynamical variables, Mean-field approximation drastically reduces this high dimensionality and the behavior of the system down to just a few dimensions by reformulating the dynamics of the system in terms of the "average state" of the system. An auxiliary field is a non-physical field which is generally added to a QFT Lagrangian to impose some constraints: dynamical or otherwise. Auxiliary quantum field is not a dynamical field. It lacks a kinetic energy term. One can identify whether a given field is auxiliary or not just by looking at the Lagrangian. If a given field's kinetic energy term is missing, it is an auxiliary field and therefore a non-propagating field. For example in a scalar quantum field, auxiliary field cancels the quadratic terms in the potential. Since, auxiliary fields are non-propagating and therefore adding these fields by hand does not change the content of the Lagrangian. It is interesting to note that the effect of an auxiliary field is the same both classically and quantum mechanically [7].

Lagrangian is a function of the coordinate q and only its first time derivative. The time integral of the Lagrangian is called the action and is denoted by S.

$$S = \int L\, dt \tag{12}$$

Lagrangian density $\mathcal{L}$ is the function of $q, \frac{dq}{dx}, \frac{dq}{dt}, x, y, z, t$ in three dimension and the function of $q, \frac{dq}{dx}, \frac{dq}{dt}, x, t$ in 1-dimension. Four-dimensional vector q(x, y, z, t) can be used that treats time and the three spatial dimensions symmetrically and equally. Integrates ($\mathcal{L}$) over all spacetime to get the action.

$$S = \int \mathcal{L}\left(q, \frac{dq}{dx}, \frac{dq}{dt}, x, y, z\right) d^3x\, dt \tag{13}$$

The spatial volume integral of the Lagrangian density is the Lagrangian in 3D,

$$L = \int \mathcal{L}\, d^3x \tag{14}$$

The Hubbard model is of electron correlations and it will eventually lead to a better understanding of the way in which the Pauli Exclusion Principle leads to magnetism. Hubbard model, on the other hand, the Hamiltonian is now made up of two terms. The first term describes the kinetic energy of the system, parameterized by the hopping integral. The second term is the on-site interaction of strength that represents the electron repulsion. If the Hamiltonian without the contribution of the second term, simply left with the tight binding formula from regular band theory. When the second term is included, however, end up with a more realistic model that also predicts a transition from conductor to insulator as the ratio of interaction to hopping. Hamiltonian of the two dimensional Hubbard model takes the form,

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} \left(c^{+}_{i,\sigma} c_{j,\sigma} + h.c \right) + U \sum_{i} n_{i,\uparrow}\, n_{i,\downarrow} \tag{15}$$

Where $h.c$ denotes Hermitian conjugation, σ is the spin index, and symbols of the form $c^{+}_{i,\sigma}$ denote second quantisation operators creating an electron with spin σ at a lattice site i. The hopping integral between sites i and j is denoted by t_{ij}.

The Hamiltonian of the 1D Hubbard model has the following mathematical form:

$$H = -t \sum_{\langle i,j \rangle} \sum_{\sigma} \left(c^{+}_{i+1\sigma} c_{i,\sigma} + c^{+}_{i,\sigma} c_{i+1\sigma} \right) + U \sum_{i} n_{i,\uparrow}\, n_{i,\downarrow} \tag{16}$$

Symbols of the form $c^{+}_{i,\sigma}$ denote second quantization operators creating an electron with spin σ at a lattice site i, while $n_{i,\uparrow}$ is the number operator for the number of electrons at an ion on lattice site i having spin up. The 2D Hubbard model can be solved in two particular limiting cases: these are the band limit and the atomic limit. The band limit corresponds to the case U = 0, and the Hamiltonian then has the form

$$H_{U=0} = \sum_{\vec{K},\sigma} \epsilon(\vec{k})\, c^{+}_{\vec{k},\sigma} c_{\vec{k},\sigma} \tag{17}$$

and

$$\epsilon(\vec{k}) = -\sum_{i0} t_{i0} \exp(i\vec{k}.\vec{R}_i) \tag{18}$$

Where $\vec{k}$ belongs to the Brillouin zone. In the case of a 2D square lattice with lattice step (a), it can be shown that

$$\epsilon(\vec{k}) = -2t\left[\cos(k_x a) + \cos(k_y a)\right] \tag{19}$$

The atomic limit of the Hubbard model corresponds to the opposite case: $t = 0$ but $U \neq 0$. The Hamiltonian then has the form,

$$H_{t=0} = U \sum_i n_{i,\uparrow} n_{i,\downarrow} \tag{20}$$

As the hopping energy is t = 0 this limit has no interest for studies of the transport properties of the Hubbard model. The Hubbard model can be solved in each of these two limiting cases separately. Complications arise when attempting to solve it in intermediary cases, when both $t \neq 0$and $U \neq 0$ [8]. Hubbard model plays a significant role in the conductor-insulator transition of metal and magnetism of narrow band [9]. For example, Hubbard-Mott insulators [10] are a class of materials that should conduct electricity under conventional band theories, but are insulators when measured (particularly at low temperatures). This effect is due to electron-electron interactions in narrow bands which are not considered in conventional band theory. The Coulomb repulsions of semi-localized electrons in narrow bands are strong enough to split the band into two sub-bands and generate the Mott band gap.

In s-wave superconductors (SCs) where the quasiparticle excitation spectrum is fully gapped and has a constant sign of the pairing form factor across the Fermi surface, Anderson's magic theorem keeps the critical temperature (T_c) robust to weak, non-magnetic impurities. In higher angular momentum SCs (p-,d-wave etc.) or SCs where the sign of the gap changes across parts of the Fermi surface (such as s± pnictide SCs), T_c is drastically suppressed with the addition of impurities–magnetic or otherwise. These effects hold in the independent disorder limit and in the absence of electron correlations [11]. To be more precise, superconductivity is characterized by an order parameter, that describes pairing of fermions with time-reversed momenta, $\boldsymbol{k}$ and $-\boldsymbol{k}$,

$$\psi(\boldsymbol{k})_{\alpha,\beta} = \langle \psi_{k,\alpha}\psi_{-k,\beta}\rangle \tag{21}$$

Where, α, β are spin indices of the paired fermionic states. The difference between the spin singlet pairing (total spin of the pair $S = 0$), for which$\psi(\boldsymbol{k})_{\alpha,\beta} = \psi(\boldsymbol{k})\left(i\sigma^y_{\alpha\beta}\right)$, where $\sigma(y)$ is the Pauli matrix in spin space,

and spin triplet state($S = 1$), when $\psi_{\alpha\beta}$ is a symmetric spinor in α, β. Since the order parameter has to be antisymmetric with respect of permutation of fermion operators, the spatial part of $\psi(\boldsymbol{k})_{\alpha,\beta}$ is even for spin singlet superconductors and odd in the spin-triplet case. Expanding in eigenfunctions of orbital momentum, it follows that spin singlet pairing corresponds to even orbital function of momentum $\boldsymbol{k}$ and hence we call it s- (for $l = 0$), d-wave (for $l = 2$), etc. superconductor in analogy with the notation for the atomic states. For spin triplet superconductor, the orbital part is an odd function of k, and hence spin triplet superconductor can be p-wave ($l = 1$), f-wave ($l = 3$) etc. [12]. The electronic Raman response is calculated in the gauge invariant form assuming that the quasiparticles interact through the long-range Coulomb potential$\left(V_q = \frac{4\pi e^2}{q^2}\right)$. The superconductivity-induced changes in the self-energy of optical, zone-center (q = 0) phonons [13].

3. Methodology

We consider the system of ultra-cold fermionic systems which is described by the three dimensional attractive Hubbard model is,

$$H = \sum_{K\sigma} \varepsilon_k c^{\dagger}_{k\sigma} c_{k\sigma} - U_R \sum_{i} c^{\dagger}_{i\uparrow} c^{\dagger}_{i\downarrow} c_{i\downarrow} c_{i\uparrow} \qquad (22)$$

Where $U_R > 0$, $c_{k\sigma}, c_{i\sigma}(c^{\dagger}_{k\sigma}, c^{\dagger}_{i\sigma})$ denote annihilation (creation) operator of a spin-σ fermion with momentum k (at site i) and ε_k is the single-particle energy dispersion. Where $\varepsilon_k = \xi_k - \varepsilon_F$ is the electron dispersion ξ_k measured relatively to the Fermi energy (ε_F). The ε_k is defined as, for U = 0 and d = 1 chain with $\varepsilon_k = -2\,t\,cosk$ and for U = 0 in two dimensional Hubbard model on a square lattice, $\varepsilon_k = -2t(\cos k_x + \cos k_y)$. If the system is subjected to inelastic collisions than scattered atoms are lost to a surrounding environment. So, the dissipation occurs in the system which is also verified from experimentally. Here, we discuss the time evolution of the density matrix ρ which is described by the Linblad master equation.The scattering atoms are lost in to an environment if the system is subjected to the inelastic collision and results dissipative dynamics. The time evolution of the density matrix ρ is described by lindblad equation which is given by [14],

$$\frac{d\rho}{dt} = \mathcal{L}\rho = -i[H,\rho] + D(\rho) \qquad (23)$$

Second part of equation (23) represents a Lindblad dissipator and $\mathcal{L}$ is Liovillian superoperator and

$$D(\rho) = \frac{\gamma}{2}\sum_i \left(L_i^\dagger \rho L_i + \rho L_i^\dagger L_i - 2L_i \rho L_i^\dagger\right) \tag{24}$$

The operators L_i are usually referred to as jump operators. On solving (23) and (24), we obtain the Liouville-von Neumann equation [15]. Action is define for open quantum system in term of Lindblad equation as

$$S = \int_\infty^\infty dt \sum_{i\alpha} \alpha \bar{c}_{i\alpha} i\partial_t c_{i\alpha} - \int_\infty^\infty dt i\mathcal{L} \tag{25}$$

Now from equation (23) and (24) using commutative relation for k-sapce we get,

$$S = \int_{-\infty}^{\infty} dt\, (\bar{c}_{k\sigma+} i\partial_t c_{k\sigma+} - \bar{c}_{k\sigma-} i\partial_t c_{k\sigma-}) - H_+ + H_- + \int_\infty^\infty dt \frac{\gamma i}{2}\sum_i (\bar{L}_{i+}L_{i+} + \bar{L}_{i-}L_{i-} - 2L_{i+}\bar{L}_{i-}) \tag{26}$$

Where the subscripts + and – denote forward and backward paths and Hamiltonian is given as,

$$H_+ = \sum_{k\sigma} \varepsilon_k \bar{c}_{k\sigma+} c_{k\sigma+} - U_R \sum_i \bar{c}_{k\uparrow+}\bar{c}_{k\downarrow+} c_{k\downarrow+} c_{k\uparrow+} \tag{27}$$

$$H_- = \sum_{k\sigma} \varepsilon_k \bar{c}_{k\sigma-} c_{k\sigma-} - U_R \sum_i \bar{c}_{k\uparrow-}\bar{c}_{k\downarrow-} c_{k\downarrow-} c_{k\uparrow-} \tag{28}$$

$$L_{i+} = c_{k\downarrow+}c_{k\uparrow+}, L_{i-} = c_{k\downarrow-}c_{k\uparrow-} \text{and } \bar{L}_{i+} = c_{k\uparrow+}c_{k\downarrow+}, \bar{L}_{i-} = c_{k\uparrow-}c_{k\downarrow-} \tag{29}$$

Using these relations in integrand of equation (26) and solving, we get the action can be written as,

$$S = \int_{-\infty}^{\infty} dt\, [\sum_{k\sigma} (\bar{c}_{k\sigma+}(i\partial_t - \varepsilon_k)c_{k\sigma+} - \bar{c}_{k\sigma-}(i\partial_t - \varepsilon_k)c_{k\sigma-}) + \sum_{kk'} (U\, \bar{c}_{k\uparrow+}\bar{c}_{k\downarrow+}c_{k\downarrow+}c_{k\uparrow+} - U^* \bar{c}_{k\uparrow-}\bar{c}_{k\downarrow-}c_{k\downarrow-}c_{k\uparrow-} - i\gamma c_{k\downarrow+}c_{k\uparrow+}\bar{c}_{k\uparrow-}\bar{c}_{k\downarrow-})] \tag{30}$$

Where, $U = U_R + \frac{i\gamma}{2}$ and $U^* = U_R - \frac{i\gamma}{2}$. Now perform the Hubbard-Stratonovich transformation for each term in the second line of equation (30) with auxiliary fields Δ_α ($\alpha = +, -, \pm$) as,

$$iU\bar{c}_{k\uparrow+}\bar{c}_{-k\downarrow+}c_{-k'\downarrow+}c_{k'\uparrow+} \rightarrow -i\Delta_+\bar{c}_{k\uparrow+}\bar{c}_{k\downarrow+} - i\bar{\Delta}_+c_{-k\downarrow+}c_{k\uparrow+} + \frac{\bar{\Delta}_+\Delta_+}{iU} \quad (31)$$

$$iU^*\bar{c}_{k\uparrow-}\bar{c}_{-k\downarrow-}c_{-k'\downarrow-}c_{k'\uparrow-} \rightarrow i\Delta_-\bar{c}_{k\uparrow+}\bar{c}_{k\downarrow+} + i\bar{\Delta}_-c_{-k\downarrow+}c_{k\uparrow+} - \frac{\bar{\Delta}_-\Delta_-}{iU^*} \quad (32)$$

$$\gamma c_{-k\downarrow+}c_{k\uparrow+}\bar{c}_{k'\uparrow-}\bar{c}_{-k'\downarrow-} \rightarrow -i\Delta_\pm\bar{c}_{k\uparrow+}\bar{c}_{k\downarrow+} - i\bar{\Delta}_\pm c_{-k\downarrow+}c_{k\uparrow+} + \frac{\bar{\Delta}_\pm\Delta_\pm}{\gamma} \quad (33)$$

Using equation (31), (32) and (33) in equation (30) and solving we get

$$S = \int dt\Bigg\{\sum_k\Big[\bar{\psi}^t_{k+}\begin{pmatrix} i\partial_t - \varepsilon_k & -\Delta_+ \\ -\bar{\Delta}_+ + i\Delta_\pm & -i\partial_t + \varepsilon_k\end{pmatrix}\psi_{k+} - \bar{\psi}^t_{k-}\begin{pmatrix} i\partial_t - \varepsilon_k & -\Delta_- - i\Delta_\pm \\ -\bar{\Delta}_- & -i\partial_t + \varepsilon_k\end{pmatrix}\psi_{k-}\Big] + \frac{\bar{\Delta}_+\Delta_+}{iU} - \frac{\bar{\Delta}_-\Delta_-}{iU^*} - \frac{\bar{\Delta}_\pm\Delta_\pm}{\gamma}\Bigg\} \quad (34)$$

Where $\bar{\psi}_{k\alpha} = (\bar{c}_{k\uparrow\alpha}, c_{-k\downarrow\alpha})^t$, $\psi_{k\alpha} = (c_{k\uparrow\alpha}, \bar{c}_{-k\downarrow\alpha})^t$ $(\alpha = +, -)$. Appling saddle-point condition $\langle\partial S/\partial\Delta_\alpha\rangle = \langle\partial S/\partial\bar{\Delta}_\alpha\rangle = 0$ $(\alpha = +, -, \pm)$ and solving equation (34) we get,

$$S = \int dt\sum_k\Big\{\bar{\psi}^t_{k+}\begin{pmatrix} i\partial_t - \varepsilon_k & -\Delta_+ \\ -\bar{\Delta}_+ + i\Delta_\pm & -i\partial_t + \varepsilon_k\end{pmatrix}\psi_{k+} - \bar{\psi}^t_{k-}\begin{pmatrix} i\partial_t - \varepsilon_k & -\Delta_- - i\Delta_\pm \\ -\bar{\Delta}_- & -i\partial_t + \varepsilon_k\end{pmatrix}\psi_{k-}\Big\} \quad (35)$$

Where the superfluid order parameter of the system is given by

$$\Delta = -\frac{U}{N_0}\sum_k tr(c_{-k\downarrow}c_{k\uparrow}\rho) \equiv -\frac{U}{N_0}\sum_k\langle c_{-k\downarrow}c_{k\uparrow}\rangle \quad (36)$$

3.1.3.1 For equilibrium State (at Finite temperature):

Let us first consider the case for zero temperature T = 0. Here, the mean-field ground state can be written with the help of the factor u_k and $v_k$as,

$$|\psi\rangle = \prod_{k<k_F}(u_k + v_kc^\dagger_{k\uparrow}c^\dagger_{-k\downarrow})\,|0\rangle \quad (37)$$

Hereby, the factor u_k describes the unoccupation of a pair state (k↑, −k↓), whereas the factor $v_k$describes the occupation of a pair state. In the normal

state, there is a sharp transition from zero to one at k_F for v_k^2 and from one to zero to one at u_k^2. In the BCS state, this transition is smeared out. The equilibrium expectation values for the electron operators can be evaluated and read [16],

$$\langle c_{-k\downarrow}c_{k\uparrow}\rangle = \langle 0|\left(\prod_{k<k_F}(u_{k'}^* + v_{k'}^* c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger)\right)c_{-k\downarrow}c_{k\uparrow}\left(\prod_{k<k_F}(u_{k'} + v_{k'}c_{k'\uparrow}^\dagger c_{-k'\downarrow}^\dagger)\right)|0\rangle \quad (38)$$

$$\langle c_{-k\downarrow}c_{k\uparrow}\rangle = u_k^* v_k = \frac{\Delta_k}{2E_k} \quad (39)$$

Similarly we get

$$\langle c_{k\uparrow+}^\dagger c_{-k\downarrow+}^\dagger\rangle = u_k v_k^* = \frac{\Delta_k^*}{2E_k} \quad (40)$$

Where, $|u_k^2| + |v_k^2| = 1$ and this constrain is conveniently imposed by letting, $u_k = sin\theta_k$ and $v_k = cos\theta_k$. Since, the BCS ground state is the vacuum state for the quasi-particle operators (γ_k). These γ_k describe the elementary quasi-particle excitations of the system, which are often called Bogoliubons. The energies of these excitations (i.e. quasi-particle excitation energy) is $E_k = \sqrt{{\varepsilon_k}^2 + {\Delta_k}^2}$, where Δ_k plays the role of an energy gap of minimum excitation energy. We work with quasi-particle creation and annihilation operators ($\gamma^\dagger, \gamma$) rather than the electron operator (c^+, c). Using Bogoliubov transformations we get,

$$\langle c_{-k\downarrow}c_{k\uparrow}\rangle = u_k v_k\left(1 - \gamma_{k0}^\dagger\gamma_{k0} - \gamma_{k1}^\dagger\gamma_{k1}\right) = u_k v_k\left(1 - f(E_k)\right) \quad (41)$$

We have identified E_k as the excitation energy of a fermion quasi-particle, it must be a positive quantity. The probability that it is excited in thermal equilibrium is the usual Fermi function

$$f(E_k) = \frac{1}{e^{\beta E_k} + 1} \quad (42)$$

Where, $\beta = \frac{1}{K_B T}$ and $E_k \geq \Delta$, $f(E_k)$goes to zero at T = 0 for all k, including $|k| < k_F$. Using equation (41) and equation (42) and solving we get

$$\langle c_{-k\downarrow} c_{k\uparrow} \rangle_T = \frac{\Delta_k}{2E_k} \tanh\left(\frac{E_k}{2K_B T}\right) \tag{43}$$

$$\langle c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} \rangle_T = \frac{\Delta_k^{*}}{2E_k} \tanh\left(\frac{E_k}{2K_B T}\right) \tag{44}$$

One can see that these expectation values follow the same symmetry as the energy gap. In Particular, if we consider $\langle c_{-k\downarrow} c_{k\uparrow} \rangle$ at k values far from the Fermi level, where$\varepsilon_k \gg \Delta$, we can approximate forT $= 0$,

$$\langle c_{-k\downarrow} c_{k\uparrow} \rangle \approx \frac{\Delta_k}{2\varepsilon_k} f_k \propto f_k \tag{45}$$

Thus, $\langle c_{-k\downarrow} c_{k\uparrow} \rangle$ at values far from k_F is a good representation of the gap symmetry and even the exact functional form. To push the system out of equilibrium in an asymmetric way, we perform a quench by changing the symmetry of these expectation values. In order to have a controlled excitation, we assume that the system starts at an equilibrium position of a different symmetry (f_k^{mod}) and let it then evolve in the system with the original symmetry (f_k). Now, define the quenched expectation values [17],

$$\langle c_{-k\downarrow} c_{k\uparrow} \rangle^q = \frac{\Delta f_k^{mod}}{2E_k^{mod}} \tanh\left(\frac{E_k^{mod}}{2k_B T}\right) \tag{46}$$

$$\langle c^{\dagger}_{k\uparrow} c^{\dagger}_{-k\downarrow} \rangle^q = \frac{\left(\Delta f_k^{mod}\right)^{*}}{2E_k^{mod}} \tanh\left(\frac{E_k^{mod}}{2k_B T}\right) \tag{47}$$

Where, $E_k^{mod} = \sqrt{\varepsilon_k{}^2 + |\Delta f_k^{mod}|^2}$, for the modified symmetry function (f_k^{mod}), we assume that it is a deviation from the original symmetry of strength g in another symmetry channel (f_k'), i.e.

$$f_k^{mod} = f_k + g f_k' \tag{48}$$

$f_k = f(\varphi)$, the symmetry function depends on the considered system. For s-wave, $f(\varphi) = 1$, For, d-wave with $f(\varphi) = cos(2\varphi)$ and for $g-$ wave $f(\varphi) = \sin(4\varphi)$, g is quenching strength whose value ranges from 0 to 1. In principle, we can choose any modified symmetry function (f_k^{mod}) to quench the system and push it out of equilibrium, for s-wave as

$$f^{mod}_{k(s-wave)} = f_k + g f' = 1 \tag{49}$$

Therefore for s-wave from equation (46) and equation (47) we have,

$$\langle c_{-k\downarrow}c_{k\uparrow}\rangle^{q(s-wave)} = \frac{\Delta f^{mod}_{k(s-wave)}}{2E^{mod}_{k(s-wave)}}\tanh\left(\frac{E^{mod}_{k(s-wave)}}{2k_BT}\right) \quad (50)$$

$$\langle c^{\dagger}_{k\uparrow}c^{\dagger}_{-k\downarrow}\rangle^{q(s-wave)} = \frac{\left(\Delta f^{mod}_{k(s-wave)}\right)^{*}}{2E^{mod}_{k(s-wave)}}\tanh\left(\frac{E^{mod}_{k(s-wave)}}{2k_BT}\right) \quad (51)$$

Where, $E^{mod}_{k(s-wave)} = \sqrt{\varepsilon_k{}^2 + |\Delta f^{mod}_{k(s-wave)}|^2}$. In principle, we can choose any modified symmetry function (f^{mod}_k) to quench the system and push it out of equilibrium, for d-wave is

$$f^{mod}_{k(d-wave)} = f_k + gf' = cos(2\varphi) + g \quad (52)$$

Hereby, $f_k = cos(2\varphi)$ is the gap symmetry and $f'_k = 1$ the quench symmetry. Also for d-wave from equation (46) and equation (47) we have,

$$\langle c_{-k\downarrow}c_{k\uparrow}\rangle^{q(d-wave)} = \frac{\Delta f^{mod}_{k(d-wave)}}{2E^{mod}_{k(d-wave)}}\tanh\left(\frac{E^{mod}_{k(d-wave)}}{2k_BT}\right) \quad (53)$$

$$\langle c^{\dagger}_{k\uparrow}c^{\dagger}_{-k\downarrow}\rangle^{q(d-wave)} = \frac{\left(\Delta f^{mod}_{k(d-wave)}\right)^{*}}{2E^{mod}_{k(d-wave)}}\tanh\left(\frac{E^{mod}_{k(d-wave)}}{2k_BT}\right) \quad (54)$$

Where, $E^{mod}_{k(d-wave)} = \sqrt{\varepsilon_k{}^2 + |\Delta f^{mod}_{k(d-wave)}|^2}$. In principle, we can choose any modified symmetry function f^{mod}_k to quench the system and push it out of equilibrium, for g-wave is

$$f^{mod}_{k(g-wave)} = f_k + gf' = sin(4\varphi) + g \quad (55)$$

Hereby, $f_k = sin(4\varphi)$ is the gap symmetry and $f'_k = 1$ the quench symmetry. Also for g-wave from equation (46) and equation (47) we have,

$$\langle c_{-k\downarrow}c_{k\uparrow}\rangle^{q(g-wave)} = \frac{\Delta f^{mod}_{k(g-wave)}}{2E^{mod}_{k(g-wave)}}\tanh\left(\frac{E^{mod}_{k(g-wave)}}{2k_BT}\right) \quad (56)$$

$$\langle c^{\dagger}_{k\uparrow}c^{\dagger}_{-k\downarrow}\rangle^{q(g-wave)} = \frac{\left(\Delta f^{mod}_{k(g-wave)}\right)^{*}}{2E^{mod}_{k(g-wave)}}\tanh\left(\frac{E^{mod}_{k(g-wave)}}{2k_BT}\right) \quad (57)$$

Where, $E^{mod}_{k(g-wave)} = \sqrt{\varepsilon_k{}^2 + |\Delta f^{mod}_{k(g-wave)}|^2}$

3.2. Pair correlation function for excitation quasiparticles in non-equilibrium

For the case of spin-singlet pairing, the pair-correlation function can be expressed by can be obtained as [18],

$$\rho = \langle c_{k\uparrow}^{\dagger} c_{-k\downarrow}^{\dagger} \rangle \langle c_{-k\downarrow} c_{k\uparrow} \rangle \tag{58}$$

The pair-correlation function, for s-wave from equation (58), equation (50) and equation (51) can be written as

$$\rho_s = \frac{(\Delta f_{k(s-wave)}^{mod})^* \Delta f_{k(s-wave)}^{mod}}{4\left(E_{k(s-wave)}^{mod}\right)^2} \left[\tanh\left(\frac{E_{k(s-wave)}^{mod}}{2k_B T}\right)\right]^2 \tag{59}$$

The pair-correlation function, for d-wave from equation (58), equation (53) and equation (54) can be written as

$$\rho_d = \frac{\left(\Delta f_{k(d-wave)}^{mod}\right)^* \Delta f_{k(d-wave)}^{mod}}{4\left(E_{k(d-wave)}^{mod}\right)^2} \left[\tanh\left(\frac{E_{k(d-wave)}^{mod}}{2k_B T}\right)\right]^2 \tag{60}$$

The pair-correlation function, for g-wave from equation (58), equation (56) and equation (57) can be written as

$$\rho_g = \frac{\left(\Delta f_{k(g-wave)}^{mod}\right)^* \Delta f_{k(g-wave)}^{mod}}{4\left(E_{k(g-wave)}^{mod}\right)^2} \left[\tanh\left(\frac{E_{k(g-wave)}^{mod}}{2k_B T}\right)\right]^2 \tag{61}$$

4. Result and Discussion

4.1. Expectation value of annihilation and creation at zero temperature

Equation (39) and (40) represent the annihilation and creation expectation value at zero temperature. The developed equation is computed for annihilation and creation considering gap energy for annihilation is1meV, creation is 1.5meV, Boltzmann constant is $8.617 \times 10^{-5}\text{eVK}^{-1}$. The nature of developed equation is computed as shown in figure 6 for annihilation and creation. The expectation of quasiparticle for creation is greater than annihilation. The expectation value decrease with increasing the excitation energy, the maximum expectation value for creation is observed at 2meV while annihilation is observed at 0.38 and the expectation value is observed minimum at 9meV.

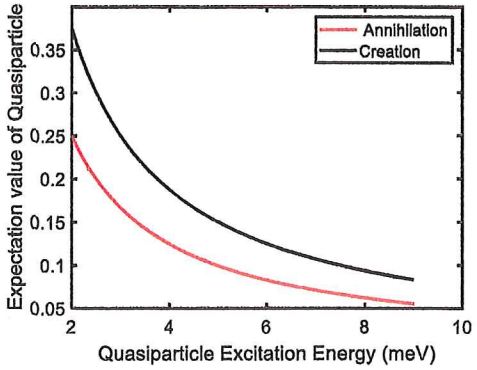


Figure 1: Excitation energy vs Expectation value at zero temperature

4.2.Expectation value of annihilation and creation operators at finite temperature (Quasi-Particles)

Equation (43) and (44) represent the annihilation and creation expectation value with temperature. The developed equation is computed for annihilation and creation considering gap energy for annihilation is1meV, creation is 1.5meV, quasiparticle excitation energy is taken as 4meV, Boltzmann constant $8.617 \times 10^{-5} eVK^{-1}$. The nature of developed equation is computed as shown in figure 7 for annihilation and creation. The expectation of quasiparticle for creation is greater than annihilation. The expectation value beyond 5K is constant and then decrease with increasing temperature, the maximum expectation value for creation and annihilation in region 1K to 5K is observed about 0.190 and 0.123, respectively. The nature of expectation value with temperature and with temperature is different for this one can visualized and compare figure 1 and figure 2.

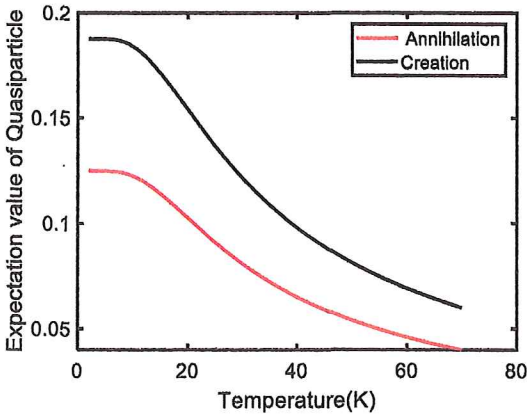


Figure 2: Excitation energy vs Expectation value with temperature

4.3.Quenching expectation value/modified value of annihilation and creation at finite temperature for s-wave

Equation (50) and (51) represent the annihilation and creation expectation value with temperature for s-wave. The developed equation is computed for annihilation and creation considering gap energy for annihilation is 1meV, creation is 1.5meV, fermi energy is taken as 4meV, Boltzmann constant $8.617 \times 10^{-5}\text{eVK}^{-1}$. The nature of developed equation is computed as shown in figure 3 for annihilation and creation with considering a function with quenching symmetry zero and gap symmetry 1. The expectation of quasiparticle for creation is greater than annihilation. The expectation value beyond 5K is constant and then decrease with increasing temperature, the maximum expectation value for creation and annihilation in region 1K to 5K is observed about 0.170 and 0.130, respectively. To study the s-wave the excitation quasiparticles energy get modified by using symmetry function (gap symmetry and quenching symmetry).

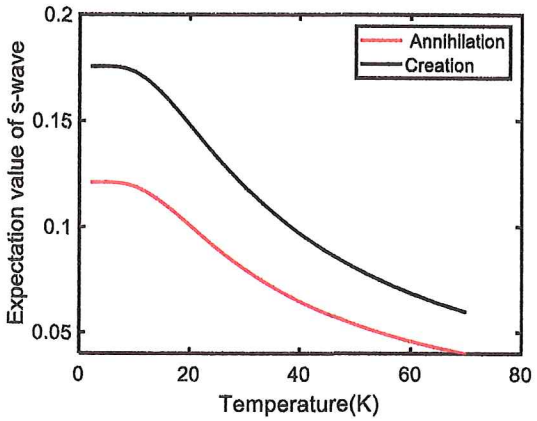


Figure 3: Excitation energy vs Expectation value with temperature for s-wave

4.4.Quenching expectation value/modified value of annihilation and creation at finite temperature for d-wave

Equation (53) and (54) represent the annihilation and creation expectation value with temperature for d-wave. The developed equation is computed for annihilation and creation considering gap energy for annihilation is1meV, creation is 1.5meV, fermi energy is taken as 4meV, Boltzmann constant $8.617 \times 10^{-5}\text{eVK}^{-1}$. The nature of developed equation is computed as shown in figure 4 for annihilation and creation with considering a function with quenching symmetry one and gap symmetry$cos(2\varphi)$. The expectation of quasiparticle for creation is greater than annihilation. The expectation value beyond 5K is constant and then decrease with increasing temperature,

the maximum expectation value for creation and annihilation in region 1K to 5K is observed. The annihilation and creation expectation value was observed at $= 0, \frac{\pi}{4}, \frac{\pi}{6}, \frac{\pi}{8}$ .The computation shows that the maximum annihilation and creation expectation value is observed as 0.170 and 0.245, respectively at $\varphi = 0$. In addition, Similar as s-wave, excitation quasiparticles energy get modified by using symmetry function (gap symmetry and quenching symmetry) for d-wave.

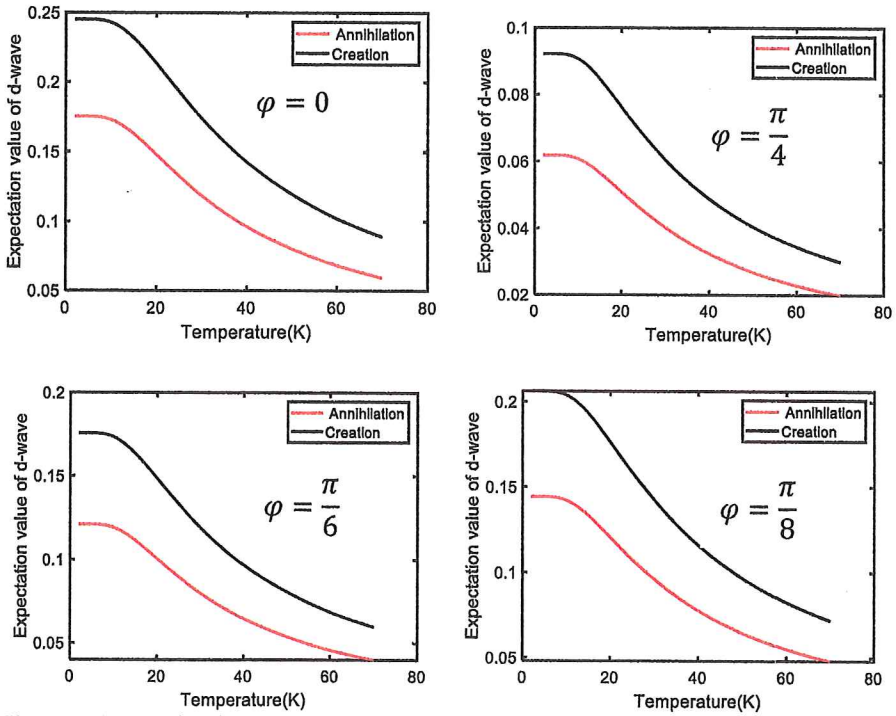


Figure 4: Excitation energy vs Expectation value with temperature for d-wave

4.5.Quenching expectation value/modified value of annihilation and creation at finite temperature for g-wave

Equation (56) and (57) represent the annihilation and creation expectation value with temperature for g-wave. The developed equation is computed for annihilation and creation considering gap energy for annihilation is1meV, creation is 1.5meV, fermi energy is taken as 4meV, Boltzmann constant $8.617 \times 10^{-5}\text{eVK}^{-1}$. The nature of developed equation is computed as shown in figure 5 for annihilation and creation with considering a function with quenching symmetry one and gap symmetry $\sin(4\varphi)$. The expectation of quasiparticle for creation is greater than annihilation. The expectation value beyond 5K is constant and then decrease with increasing temperature,

the maximum expectation value for creation and annihilation in region 1K to 5K is observed. The annihilation and creation expectation value was observed at $= 0, \frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{7}$in figure 5, respectively. The computation shows that the maximum annihilation and creation expectation value is observed as 0.170 and 0.245, respectively at $\varphi = \frac{\pi}{7}$. In addition, Similar as s-wave and d-wave, excitation quasiparticles energy get modified by using symmetry function (gap symmetry and quenching symmetry) for g-wave.

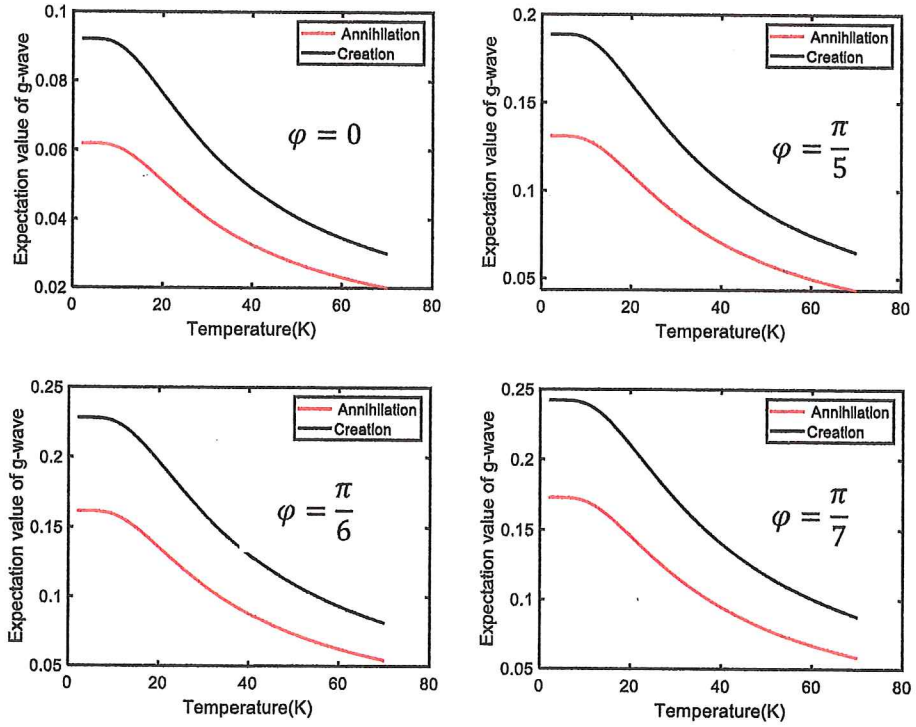


Figure 5: Excitation energy of quasiparticles vs Expectation value with temperature for d-wave

4.6.Correlation function of spin-singlet pairing for s, d and g −wave

The pair correlation function for annihilation and creation was also developed for s, d and g-wave in equation (59), (60) and (61), respectively. The developed equation is computed considering gap energy for annihilation is1meV, creation is 1.5meV, fermi energy is taken as 4meV, Boltzmann constant $8.617 \times 10^{-5}\text{eVK}^{-1}$. The gap symmetry function for s-wave, d-wave and g-wave is different and energy of quasiparticles depend upon on it. Therefore, the expectation value get modified and hence the pair correlation function depend on gap symmetry function, fermi energy, quasiparticle excitation energy, more detail in equation (59), (60) and (61). The computed

of pair correlation function with temperature at different angle is shown in figure 6.

The pair correlation function of quasiparticle beyond 5K is constant and then decrease with increasing temperature, the maximum pair correlation function is observed in region 1K to 5K. The pair correlation function was observed at $\varphi = 0, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{5}, \frac{\pi}{6}, \frac{\pi}{7}$ in figure 6, respectively. The computation shows that the maximum pair correlation annihilation function is observed as 0.048 at $\varphi = 0$ for d-wave. At $\varphi = \frac{\pi}{2}$ the pair correlation function of d-wave and g-wave is same, at $\varphi = \frac{\pi}{3}, \frac{\pi}{6}$ the pair correlation function of d-wave is absence.

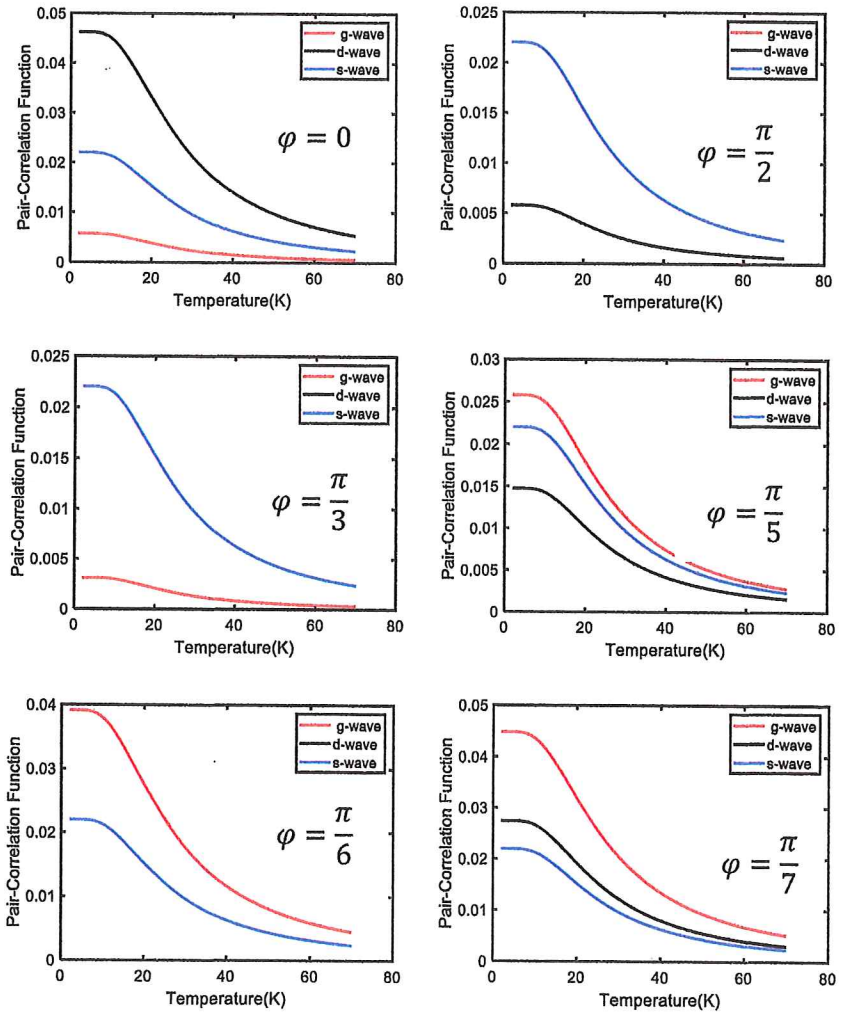


Figure 6: Excitation energy vs Expectation value with temperature for g-wave

4. Conclusion

The development equation to study the expectation value of excitation quasiparticle for s-wave, d-wave and g-wave shows, expectation value decrease with increasing the temperature for annihilation and creation. In addition, the expectation value for annihilation for all case found less than creation. The pair correlation function for same quasiparticles for s-wave, d-wave and g-wave also decrease with temperature. In some case the pair function correlation is absence like $\varphi = \frac{\pi}{3}, \frac{\pi}{6}$ the pair correlation function of d-wave is absence. The expectation value of excitation quasiparticles inversely proportional to excitation quasiparticles energy, directly proportional to gap, quenching symmetry function, gap energy and tanh. Similarly, pair correlation function is inversely proportional to square of excitation quasiparticle energy, directly proportional to gap, quenching symmetry function, but for gap energy and tanh proportional with square.

Acknowledgment

We would like to thanks all the faculties' member of Department of Physics, Tribhuvan University, Nepal; Innovative Ghar Nepal and Robotics Academy of Nepal for their support during this work.

References

[1] A. Ishihara, Condensed matter physics, Dover publication (First Edition), (2007).

[2] J. F. Annett, Superconductivity, Super fluids and Condensates, Oxford University Press (First Edition, (2004).

[3] M. Tinkham, *Introduction to Superconductivity*, McGraw Hill, New York, (1996).

[4] K. E. Gray, Non equilibrium Superconductivity, Phonons and Kapitza Boundaries, NATO ASI Series: Plenum, New York, (1981).

[5] J.V. Neumann, Mathematische Grundlagen der Quantummechanik, Springer, Berlin, (1932).

[6] E. Schrodinger, Ann. der Phy. 79 (1926).

[7] D. Sherrington, J. Phys. C: Solid State Phys., 4, 4 (1971).

[8] V. Celebonovic, J. Phys.: Conf. Series, 253, (2010).

[9] J. Hubbard, Phys. Rev. B, 19, 5, (1979).

[10] N. F. Mott and R. Peierls, Proc. Phys. Soc. A, 49, 72, (1937).

[11] C. Setty, Phys. Rev. B, 99, 14, (2019).

[12] A. V. Balatsky, I. Vekhter and J. X. Zhu, Rev. Mod. Phys. 78, (2006).

[13] A. P. Schnyder and C. Mudry, D. Manske, Rev. Mod. Phys. 75, (2007).

[14] C. W. Gardiner and P. Zoller, *Quantum Noise*, Springer, Berlin, (2000).

[15] H. P. Breuer and F. Petruccione, The Theory of Open Quantum Systems, Oxford University Press, (2002).

[16] L. Schwarz, B. Fauseweh, N. Tsuji, and et al., *Nat. Commun.* 11, (2020).

[17] National Research Council, Condensed-Matter and Materials Physics: Basic Research for Tomorrow's Technology, The National Academies Press, Washington, DC, (1999).

[18] C. C. Tsuei and J. R. Kirtley, Rev. Mod. Phy. 72, 4, (2000).

HADRONIC JOURNAL 45, 81 - 95 (2022)
DOI 10.29083/HJ.45.01.2022

KEPLER PROBLEM AND PERIHELION IN THE MODEL OF THE DYNAMIC QUANTUM VACUUM

Davide Fiscaletti
SpaceLife Institute, Via Roncaglia, 35 – 61047 San Lorenzo in Campo (PU), Italy
spacelife.institute@gmail.com
Štefan Čelan
Scientific Research Centre Bistra, Slovenski trg 6, SI-2250 Ptuj, Slovenia
stefan.celan@bistra.si
Amrit Šorli
Scientific Research Centre Bistra, Slovenski trg 6, SI-2250 Ptuj, Slovenia
amrit.sorli@bistra.si

Received March 14, 2022
Revised March 16, 2022

Abstract

A new treatment of the Kepler problem of the orbits of a moving body and of planetary precession inside a model three-dimensional quantum vacuum based on a variable quantum vacuum energy density is suggested. In this model, the rotation of stellar objects is also rotating the dynamic quantum vacuum which influences the motion of stellar objects in a two-body problem and is the physical cause of planets' precession in our solar system. The so-called "frame-dragging effect" of space-time that causes perihelion precession is represented as a pushing force of a rotating dynamic quantum vacuum. Differences with respect to the predictions of general relativity are analysed. Perspectives regarding what happens at the horizon of events are explored.

Key words: Kepler problem, general relativity, three-dimensional dynamic quantum vacuum, quantum vacuum energy density fluctuations.

Copyright © 2022 by Hadronic Press, Inc., Palm Harbor, FL 34684, USA

1. Introduction

In general theory of relativity, the Kepler problem regarding the determination of the orbits of a body of a mass m in the gravitational field generated by a spherical mass M, where $m \ll M$, can be solved by integrating the equation of geodesics with Schwarzschild metric, or by using the integrals of motion, namely the conservation of the angular momentum, as in the Newtonian case. Einstein faced this problem regarding the derivation of the perihelion shift $\Delta\phi$ and the orbital equation $r(\phi)$ for a bound test particle of mass m following a time-like geodesic in a Schwarzschild metric about a central mass M, in 1915 [1], by finding the equation of motion

$$\frac{1}{2}\dot{r}^2 - \frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3} - \frac{E^2+c^2}{2} = 0 \qquad (1)$$

which shows that the radial motion corresponds to the motion of a body in a dimension, r, in an effective potential

$$V = -\frac{GM}{r} + \frac{L^2}{2r^2} - \frac{GML^2}{c^2r^3}. \qquad (2)$$

This effective potential is given by the Newtonian potential plus an additional relativistic term, which produces an attractive force

$$F = -\frac{3GML^2}{c^2r^4} \qquad (3)$$

which is a magnetic-type force that becomes important at short range, whose effect is to generate a precession of the perihelion of the orbit of the body, given, at each orbit, by the angle

$$\Delta\phi \sim \frac{6\pi GM}{c^2 r_*} \qquad (4)\ [2].$$

As regards Einstein's derivation, for the specific case of Mercury's orbit around the Sun, where $GM_\odot/pc^4 \sim 10^{-8}$, by applying perturbation techniques one can easily obtain valid approximations for $\Delta\phi$ and $r(\phi)$. Instead, if the parameter $GM_\odot/pc^4$ turns out to be much larger, which occurs for elliptical orbits around a massive black hole, exact solutions for the orbital equation and the perihelion shift are required. In this regard, an exact solution for the perihelion shift $\Delta\phi$ and the orbital equation $r(\phi)$ of a bound test particle in the Schwarzschild metric, were computed in [3] by using the Weierstrass function, while in [4] in a compact form that does not explicitly use the Weierstrass function but only uses the eccentricity and the semi-latus rectum p.

In this paper, our aim is to introduce a new way of treating the Kepler problem which is based on a model of a three-dimensional (3D) dynamic quantum vacuum (DQV) – proposed by the authors in several recent works [5-11] – where the origin of gravity are the energetic fluctuations of the quantum vacuum with respect to the Planck energy density, defined as the ground state of the vacuum. The paper is so

structured. In section 2 we will make a review of the fundamental foundational ideas of the 3D DQV vacuum model. In section 3 we will provide the calculations of the perihelion shift $\Delta\phi$ and the orbital equation $r(\phi)$ for a bound test particle, in terms of the variable quantum vacuum energy density, discussing the differences with respect to the predictions of standard general relativity. Finally, in section 4 we will analyse some perspectives of this approach as regards the unstable orbits near the horizon.

2. The foundational ideas of the three-dimensional dynamic quantum vacuum model in a nutshell

According to the 3D DQV model, the fundamental starting-ideas can be so summarized. First, the ultimate background of processes is an isotropic, granular, three-dimensional (3D) "quantum vacuum" whose elementary grains are energetic packets having the size of Planck's volume. The energy density is the most universal physical property of the quantum vacuum and defines its different specific states. The ground state of the 3D quantum vacuum is defined by the Planck energy density

$$\rho_{pE}=\frac{m_p\cdot c^2}{l_p^3}=4{,}641266\cdot 10^{113}\,\frac{Kg}{ms^2} \tag{5}$$

which corresponds to the free space, namely the physical situation of space in absence of material objects. The 3D quantum vacuum is characterized by an excited state when one has the appearance of material objects and can be associated with the energy density

$$\rho=\rho_{pE}-\frac{m\cdot c^2}{V} \tag{6}$$

(and by the change of the energy density

$$\rho_{pE}-\rho\equiv\Delta\rho=\frac{m\cdot c^2}{V}\,, \tag{7}$$

with respect to the ground state), where m is the mass of the material body into consideration. In other words, in this picture, mass may be expressed as

$$m=\frac{V\cdot\Delta\rho}{c^2} \tag{8}$$

namely there is a symmetry between appearance of mass and fundamental variations of the quantum vacuum energy density. The physical consequence of the presence of material objects lies in the diminishing of the energy density of quantum vacuum. As a

consequence, gravity may be associated to fluctuations of the quantum vacuum energy density which generate a dragging, pushing effect of the 3D quantum vacuum. In particular, the pushing effect of a region of volume V of the DQV in a given point may be described by defining a velocity of the 3D DQV on the basis of equation

$$v_{qv} = \frac{2\Delta\rho_{qvE}V}{\hbar n}R \tag{9}$$

where R is the distance from the point to the centre of that volume, n is the number of the virtual sub-particles originating the vacuum fluctuations in the volume V (these virtual particles, on the basis of the treatment provided in [10, 11], can be associated to elementary RS (reduction-state) processes giving rise to a Bose-Einstein condensate).

The DQV velocity (9) is defined with respect to the special rest frame associated with the quantum vacuum energy density (5), which identifies a 3D Euclid space as a preferred fundamental arena, which is quantitatively defined by Galilean transformations for the three spatial dimensions

$$\begin{aligned} X' &= X - v\cdot\tau \\ Y' &= Y \\ Z' &= Z \end{aligned} \tag{10}$$

and Selleri's transformation

$$\tau' = \sqrt{1-\frac{v^2}{c^2}}\cdot\tau \tag{11}$$

for the rate of clocks (where, as usual, v is the velocity of the moving frame O'X'Y'Z', τ is the speed of clock of the observer in the rest frame OXYZ, τ' is the speed of clock of the moving observer). According to equation (11), the clocks' running is influenced by the motion relative to the rest frame and all material objects can be associated with regions of the DQV endowed with a velocity (9) relative to this special frame.

3. Calculations of the perihelion shift and the orbital motion of a material object in the model of the three-dimensional dynamic quantum vacuum

Although many physicists believe that the Mercury's perihelion advance has been understood in detail, indeed full exact solutions of the general relativity equations of a test particle in motion inside a gravitational field, which would reveal the dynamic structure of the curved space, have not been obtained until now. As regards the problem of orbits in general relativity, Vankov has recently introduced a sort of

generalization of special relativity particle dynamics where the static mass or the mass in the commoving reference frame are field-dependent proper mass, providing results which are in accordance with general relativity but avoiding superluminal motion in any circumstances [12].

Here, in our model, our aim is to find exact solutions of the orbits in general relativity, by developing a general picture which takes into account that the fundamental origin of gravity is the variable energy density of the 3D DQV. In this way, as regards our calculations of the Kepler orbits of a material body of mass m in a variable quantum vacuum energy density, we start from the following general equation

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{c^2L^2} - \left(1 - 2\frac{GV\Delta\rho_{qvE}}{c^5}u\right)\left(\frac{m^2c^2}{L^2} + u^2\right) \quad (12)$$

ruling the behaviour of the quantity $u = \frac{1}{r}$. In equation (12) ϕ is the azhimutal angle, $r(\phi)$ is the radial distance from the mass m to the point of the volume V characterized by the minimum value of the quantum vacuum energy density. Moreover, E is the energy of the moving body, L is the angular momentum. In particular, the energy E is determined by the initial conditions of the planet's orbit. As regards the angular momentum, it is given by

$$L = m\omega r^2. \quad (13)$$

In equation (13) ω is the proper angular speed of the planet and is associated both to the orbital velocity of the planet v_p and to the velocity of the quantum vacuum on the planet's orbit v_{qv} on the basis of relation:

$$\omega = \frac{v_p + v_{qv}}{r} \quad (14) \text{ [9].}$$

By inserting (13) and (14) into (12), this last equation becomes

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{c^2\left(m(v_p+v_{qv})r\right)^2} - \left(1 - 2\frac{GV\Delta\rho_{qvE}}{c^5}u\right)\left(\frac{m^2c^2}{\left(m(v_p+v_{qv})r\right)^2} + u^2\right) \quad (15)$$

and, taking account of the mathematical formulation of the velocity of the 3D DQV (9):

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{c^2\left(m\left(v_p+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)r\right)^2} - \left(1 - 2\frac{GV\Delta\rho_{qvE}}{c^5}u\right)\left(\frac{m^2c^2}{\left(m\left(v_p+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)r\right)^2} + u^2\right) \quad (16)$$

namely

$$\left(\frac{du}{d\phi}\right)^2 = \frac{E^2}{c^2r^2m^2\left(v_p+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^2} - \left(1 - 2\frac{GV\Delta\rho_{qvE}}{c^5}u\right)\left(\frac{c^2}{r^2\left(v_p+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^2} + u^2\right). \quad (17)$$

For bound orbits, equation (17) may be expressed as

$$\left(\frac{du}{d\phi}\right)^2 = \frac{(u_p+u_a)\left(1-2\frac{GV\Delta\rho_{qvE}}{c^5}u_p\right)\left(1-2\frac{GV\Delta\rho_{qvE}}{c^5}u_a\right)}{2\frac{GV\Delta\rho_{qvE}}{c^5}} - \left(1 - 2\frac{GV\Delta\rho_{qvE}}{c^5}u\right)\frac{\left[u_p+u_a-2\frac{GV\Delta\rho_{qvE}}{c^5}(u_p^2+u_a^2+u_pu_a)+2\frac{GV\Delta\rho_{qvE}}{c^5}u^2\right]}{2\frac{GV\Delta\rho_{qvE}}{c^5}} \quad (18)$$

where u_p and u_a are the values of u in perihelion and aphelion respectively and one has

$$\frac{E^2}{c^2r^2m^2\left(v_p+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^2} - \left(1-2\frac{GV\Delta\rho_{qvE}}{c^5}u_p\right)\left(\frac{c^2}{r^2\left(v_p+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^2}+u_p^2\right) = 0 \quad (19)$$

$$\frac{E^2}{c^2r^2m^2\left(v_p+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^2} - \left(1-2\frac{GV\Delta\rho_{qvE}}{c^5}u_a\right)\left(\frac{c^2}{r^2\left(v_p+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^2}+u_a^2\right) = 0 \quad (20)$$

since the radial speed is zero at u_p and u_a.

By rearranging equation (18), one gets

$$\left(\frac{du}{d\phi}\right)^2 = 2\frac{GV\Delta\rho_{qvE}}{c^5}(u-u_p)(u-u_a)(u-u_3) \quad (21)$$

where

$$u_3 = \frac{c^5}{2GV\Delta\rho_{qvE}} - (u_p+u_a). \quad (22)$$

In equation (21) one has the result $(u-u_p)(u-u_a) < 0 \quad \forall u \in |u_a, u_p|$ and, as a consequence, $(u-u_3)$ must be negative $\forall u \in |u_a, u_p|$. In light of this constraint, an upper limit for the variable quantum vacuum energy density follows, which is given by the following relations

$$\Delta\rho_{qvE} < \frac{c^5}{2GV(2u_p+u_a)} \quad \text{when} \quad u-u_3 < 0 \quad (23a)$$

$$\frac{c^5}{2GV(2u_p+u_a)} < \Delta\rho_{qvE} < \frac{c^5}{2GV(2u_a+u_p)} \quad \text{when} \quad -(u_p-u_a) < u-u_3 < u_p-u_a \quad (23b)$$

$$\Delta\rho_{qvE} > \frac{c^5}{2GV(2u_a+u_p)} \quad \text{when} \quad u-u_3 > 0. \quad (23c)$$

Expanding the product and making some algebra in equation (18), one finds a quantity that can be integrated:

$$\left(\frac{du}{d\phi}\right)^2 = 2\frac{GV\Delta\rho_{qvE}}{c^5}[(u-f)^2-b^2]\left[u-f-\left(\frac{c^5}{2GV\Delta\rho_{qvE}}-3f\right)\right] = 2\frac{GV\Delta\rho_{qvE}}{c^5}(y^2-b^2)(y-d) \quad (24)$$

where

$$f = \frac{u_p+u_a}{2} \quad (25)$$

$$b = \frac{u_p-u_a}{2} \quad (26)$$

$$y = u-f \quad (27)$$

$$d = \frac{c^5}{2GV\Delta\rho_{qvE}} - \frac{3}{2}(u_p + u_a). \tag{28}$$

Now, if one solves (24), the integration gives

$$\phi + k_1 = \frac{1}{\sqrt{2\frac{GV\Delta\rho_{qvE}}{c^5}}}\int \frac{dy}{\sqrt{(y^2-b^2)(y-d)}} = \frac{2}{\sqrt{2\frac{GV\Delta\rho_{qvE}}{c^5}(b+d)}} F\left(sin^{-1}\left(\sqrt{\frac{y+b}{2b}}\right), \frac{2b}{b+d}\right) \tag{29}$$

where

$$F = \int_0^{\varphi} d\theta (1 - msin^2\theta)^{-1/2} \tag{30}$$

and $m < 1$. By using equations (25)-(28), the solution for the azhimutal angle may be expressed as

$$\phi + k_1 = \frac{2}{\sqrt{2\frac{GV\Delta\rho_{qvE}}{c^5}u_*}} F\left(sin^{-1}\left(\sqrt{\frac{u-u_a}{u_p-u_a}}\right), \frac{u_p-u_a}{u_*}\right) \tag{31}$$

where k_1 is a constant of integration and

$$u_* = b + d = u_3 - u_a = \frac{1}{2\frac{GV\Delta\rho_{qvE}}{c^5}} - (2u_a + u_p). \tag{32}$$

With these ingredients, the perihelion shift is given by relation

$$\Delta\phi = 2\left[(\phi + k_1)\Big|_{\phi_a}^{\phi_p}\right] - \pi = 2\left(\sqrt{\frac{2}{\frac{GV\Delta\rho_{qvE}}{c^5}u_*}} K\left(\frac{u_p-u_a}{u_*}\right) - \pi\right), \quad \text{with } \Delta\rho_{qvE} < \frac{c^5}{2GV(2u_p+u_a)} \tag{33}$$

where

$$K = \int_0^{\frac{\pi}{2}} d\theta (1 - msin^2\theta)^{-1/2}. \tag{34}$$

Now, one can also express the formalism (33) in function of the eccentricity e, the semi-latus rectum p and length of the semi-major axis a of the orbit. In this regard, taking account of the following relations

$$u_p = \frac{1+e}{p} \tag{35}$$

$$u_a = \frac{1-e}{p} \tag{36}$$

$$\frac{2}{p} = u_p + u_a \tag{37}$$

$$e = \frac{u_p-u_a}{u_p+u_a} \tag{38}$$

$$\frac{2e}{p} = u_p - u_a \tag{39}$$

$$p = \frac{1}{2}\left(\frac{1}{u_p} + \frac{1}{u_a}\right)(1 - e^2), \tag{40}$$

the condition $\Delta\rho_{qvE} < \frac{c^5}{2GV(2u_p+u_a)}$ may be written as

$$\Delta\rho_{qvE} < \frac{pc^5}{2GV(3+e)} \tag{41}$$

and thus equation (33) may be easily expressed in terms of eccentricity, semilatus rectum and length of the semi-major axis of the orbit. In particular, by making an expansion for $\Delta\rho_{qvE} \ll \frac{pc^5}{2GV}$, one finds

$$\Delta\phi = \frac{6GV\Delta\rho_{qvE}\pi}{pc^5} + \frac{3\pi G^2V^2{\Delta\rho_{qvE}}^2}{2p^2c^{10}}(18+e^2) + \frac{45\pi G^3V^3{\Delta\rho_{qvE}}^3}{2p^3c^{15}}(6+e^2) + \cdots (42)$$

The mathematical formulation (42) of the perihelion shift of a planet may be equivalently expressed in terms of the velocity of the quantum vacuum in the form

$$\Delta\phi = \frac{3\hbar n v_{qv} G\pi}{Rpc^5} + \frac{3\pi G^2 {v_{qv}}^2\hbar^2 n^2}{8p^2c^{10}}(18+e^2) + \frac{45\pi\hbar^3 n^3 G^3 {v_{qv}}^3}{16p^3c^{15}}(6+e^2) + \cdots \quad (43)$$

As regards the result (42) (or the equivalent (43)), one may note that the first term is in good agreement with the standard result obtained with perturbation techniques as well as inside the original Einstein's treatment, where it is $\Delta\phi \sim \frac{6\pi GM}{c^2 r_*}$, r_* being is the local minimum of the potential (2). The correction provided by equation (42) of our model, based on a treatment in terms of the variable energy density of the 3D DQV, turns out to be of the kind $1{,}19 \cdot 10^{-7}$, due to the fact that for the perihelion shift of Mercury one has $M_\odot/p \sim 10^{-8} \ll 1$, $M_\odot$ being the mass of the Sun. Instead, the exact result (42) shows that a treatment in terms of the variable quantum vacuum energy density is advantageous in situations when $\Delta\rho_{qvE}/p$ is much larger. The bigger value determined by the constraint (41) is $\Delta\rho_{qvE}/p \sim 10^{-1}\frac{c^5}{GV}$ and the relative error is associated to this parameter.

In synthesis, we can say that, while in Einstein's original approach the origin of the precession of the perihelion of the orbit the force (3), which generates a circular orbit at the local minimum of the potential with frequency of oscillation around this minimum, breaks the Newtonian equality, being different with respect to the angular velocity, in our model the ultimate origin of the precession of the perihelion is represented by the variable quantum vacuum energy density and the dragging effect of the vacuum corresponding to its velocity. The velocity of the quantum vacuum at the planet's orbit is determined by the diminishing of the energy density in that region, on the basis of equation (9). Taking into account equation (9), the velocity of the quantum vacuum decreases with the increasing of the distance from the Sun. The velocity of the quantum vacuum (9), as a consequence of the variable quantum vacuum energy density, is responsible of the precession, as described by equation (43). Equation (43), united with equation (9), implies that for planets closer to the Sun, the velocity of the quantum vacuum at a planet's orbit turns out to be higher and therefore the amount of precession is higher. Instead, it predicts that for planets further from the

Sun, the velocity of the quantum vacuum at a planet's orbit turns out to be lower and therefore the amount of precession is lower [9].

On the basis of the mathematical formalism developed in this paper, which leads to the mathematical formulation (42) and (43) as regards the perihelion shift, and taking account of the elements of planetary orbits, we can construct the following table showing the values of relativistic precession and the specific velocity of the quantum vacuum for each planet (in one second) [9]:

Planet	Velocity v_{qv} of quantum vacuum in one second (Km/s)	Mean orbital velocity v_p of the planet (Km/s)	Mean distance from the Sun $(10^6\ Km)$	Mass $(24\ Kg)$	Calculation of precession per century by using $\Delta\phi = \frac{\frac{6GV\Delta\rho_{qvE}\pi}{pc^5}}{\frac{3\hbar n v_{qv} G\pi}{Rpc^5}} = (arcs)$
Mercury	39.3943	47.88	57.9	0.33	42.9195
Venus	11.2807	35.02	108.2	4.87	8.6186
Earth	5.9010	29.79	149.6	5.98	3.8345
Mars	2.5427	24.13	227.9	0.65	1.3502
Jupiter	0.2180	13.97	778.3	1900	0.0623
Saturn	0.0648	9.67	1427	570	0.0137
Uranus	0.0160	6.81	2869.6	87	0.0024
Neptune	0.0065	5.45	4496.6	100	0.0008
Pluto	0.0038	4.74	5900	0.7	0.0004

The results expressed in this table are compatible, at the first order of the expansion in equations (42) and (43), with the corresponding results of Einstein's calculations and provide them a physical explanation in terms of the behaviour of a more fundamental variable quantum vacuum

energy density and the corresponding rotation effect associated with the velocity of the quantum vacuum.

Moreover, by inverting equation (31) and by using equations (35)-(40), one can write the orbit equation and express it in terms of p and e. In particular, under the constraint $\Delta\rho_{qvE} \ll \frac{pc^5}{2GV}$, if one makes an expansion, one obtains

$$u(\phi) = \frac{1+e\cos\phi}{p} + \frac{e\frac{GV\Delta\rho_{qvE}}{c^5}\sin\phi(3\phi+e\sin\phi)}{p^2} + \cdots, \quad for \quad \Delta\rho_{qvE} < \frac{pc^5}{2GV(3+e)}. \quad (44)$$

In equation (43), the first term corresponds to the non-relativistic bound Kepler orbit for a simple $1/r^2$ radially inwards central force, while the second term is coherent with the predictions of general relativity for the bound Kepler problem in the picture of a standard perturbation method.

On the other hand, as regards angular momentum and energy of the body, by solving equations (19) and (20), one finds

$$\left(\frac{E}{m}\right)^2 = \frac{1-4\frac{GV\Delta\rho_{qvE}}{pc^5}+4\frac{G^2V^2\Delta\rho_{qvE}{}^2}{p^2c^{10}}(1-e^2)}{1-\frac{GV\Delta\rho_{qvE}}{pc^5}(3+e^2)} \quad (45)$$

and

$$\left[\left(v_p + 2\frac{V\Delta\rho_{qvE}}{\hbar n}\right)r\right]^2 = \frac{\frac{GV\Delta\rho_{qvE}}{c^5}p}{1-\frac{GV\Delta\rho_{qvE}}{pc^5}(3+e^2)}. \quad (46)$$

On the basis of equations (45) and (46), $\left(\frac{E}{m}\right)^2 > 0$ and $\left[\left(v_p + 2\frac{V\Delta\rho_{qvE}}{\hbar n}\right)r\right]^2 > 0$ under the constraint $\Delta\rho_{qvE} < \frac{pc^5}{GV(3+e^2)} < \frac{pc^5}{2GV(1+e)}$ which, being less stringent than the original $\Delta\rho_{qvE} < \frac{pc^5}{2GV(3+e)}$, is satisfied.

Finally, at the end of this section, we mention that, according to general relativity, an ideal gyroscope in orbit around the Earth undergoes two relativistic precessions with respect to a distant inertial frame, namely a geodetic drift in the orbit plane due to motion through the space-time curved by Earth's mass and a frame-dragging precession due to Earth's rotation [13], which have been confirmed by the results of the Gravity Probe B experiment [14]. If in general relativity, the precession due to the geodetic effect and the frame-dragging effect is given by Schiff's formula

$$\vec{\Omega}_{GR} = \vec{\Omega}_g + \vec{\Omega}_{fd} = \frac{3GM}{2c^2r^3}\left(\vec{r}\times\vec{v}\right) + \frac{GI}{c^2r^3}\left[\frac{3\vec{r}}{r^2}\left(\vec{S}\cdot\vec{r}\right) - \vec{S}\right] \quad (47)$$

where M, I and $\vec{S}$ are mass, moment of inertia and angular momentum of the central body while $\vec{r}$ and $\vec{v}$ are the radial position and instantaneous velocity of the test body, according to our model of 3D DQV, Schiff's formula (47) is replaced by equation

$$\vec{\Omega}_{GR}=\vec{\Omega}_{g}+\vec{\Omega}_{fd}=\frac{3GV\cdot\Delta\rho_{qvE}}{2c^{4}r^{3}}\left(\vec{r}\times\left(\vec{v}_{p}+\vec{v}_{qv}\right)\right)+\frac{GI\left(\Delta\rho_{qvE}\right)}{c^{2}r^{3}}\left[\frac{3\vec{r}}{r^{2}}\left(\vec{S}\cdot\vec{r}\right)-\vec{S}\right]. \quad (48)$$

On the basis of equation (48), the geodetic effect of general relativity is directly determined by the velocity, and thus the rotaton, of the quantum vacuum energy density (associated ultimately with the motion of the virtual particles of the RS processes characterizing the region into consideration) and, because of the indirect dependence of the moment of inertia of the central body on the changes of the quantum vacuum energy density, also the frame-dragging effect implicitly depends on the quantum vacuum energy density.

4. Perspectives about the unstable orbits near the horizon

In the model here explored, where the ultimate origin of processes is a 3D DQV, the motion of r is similar to the motion of a body in an effective potential depending of the variable energy density of the 3D DQV, given by relation:

$$V=-\frac{GV\Delta\rho_{qvE}}{c^{2}r}+\frac{m^{2}r^{2}\left(v_{p}+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^{2}}{2r^{2}}-\frac{GV\Delta\rho_{qvE}}{c^{4}r^{3}}m^{2}r^{2}\left(v_{p}+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^{2}. \quad (49)$$

This potential has a maximum and a minimum. These two extreme points occur at the values:

$$r_{\pm}=\frac{c^{2}m^{2}r^{2}\left(v_{p}+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^{2}\pm\sqrt{c^{4}m^{4}r^{4}\left(v_{p}+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^{4}-12m^{2}r^{2}\frac{G^{2}V^{2}\Delta\rho_{qvE}{}^{2}}{c^{2}}\left(v_{p}+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^{2}}}{2GV\Delta\rho_{qvE}}. \quad (50)$$

When the quantity $m\left(v_{p}+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)r$ is big, the square root is positive and the potential has two extremes. The bigger extreme is a minimum and corresponds to the stable circular Keplerian orbit analysed in the previous section. The other extreme is a maximum and is associated to another circular orbit, which is unstable. One can show that the minimum stable orbit occurs when

$$m^{2}r^{2}\left(v_{p}+2\frac{V\Delta\rho_{qvE}}{\hbar n}r\right)^{2}=12\frac{G^{2}V^{2}\Delta\rho_{qvE}{}^{2}}{c^{6}}. \quad (51)$$

By substituting (51) into equation (50), one obtains

$$r_{-}=\frac{12\frac{G^{2}V^{2}\Delta\rho_{qvE}{}^{2}}{c^{4}}-\sqrt{144\frac{G^{4}V^{4}\Delta\rho_{qvE}{}^{4}}{c^{8}}-144\frac{G^{4}V^{4}\Delta\rho_{qvE}{}^{4}}{c^{8}}}}{2GV\Delta\rho_{qvE}} \quad (52)$$

namely

$$r_{-}=\frac{12\frac{G^{2}V^{2}\Delta\rho_{qvE}{}^{2}}{c^{4}}}{2GV\Delta\rho_{qvE}} \quad (53)$$

namely

$$r_{-} = 3\frac{GV\Delta\rho_{qvE}}{c^4}. \quad (54)$$

We have thus found that there are not stable orbits when the radius is less than $r_{-} = 3\frac{GV\Delta\rho_{qvE}}{c^4}$. This result is in perfect agreement with standard Einstenian general relativity where the minimum stable orbit corresponds to three times the Schwarzschild radius

$$r_{-} = 3\frac{GM}{c^2}. \quad (55)$$

In our approach, therefore, the following suggestive scenario emerges. The ultimate origin of the existence of the minimum stable orbit which is observed in the accretion disks of black holes is represented by the fluctuations of the energy density of the 3D DQV and the resulting velocity of quantum vacuum and its rotation effect. In particular, in black holes the quantum vacuum energy density is at a critical low value which implies that, inside Schwarzschild radius, matter is unstable and transforms back into elementary particles that form AGNs jets [15].

On the other hand, in general relativity, the instability of an orbit can be associated with a positive value of a Lyapunov exponent [16, 17], whose maximal values can distinguish a chaotic dynamics from others, which may initiate unstable/homoclinic orbits by a perturbation. In particular, for a spinning black hole the number of unstable orbits increase very rapidly and crowded into the corresponding phase [18-20]. By following the treatment of Pretorius and Khurana [21], the Lyapunov exponent can be defined from equation

$$\lambda = \frac{\Omega}{2\pi\sqrt{\frac{\dot{r}^2}{2}}} \quad (56)$$

where Ω is the angular velocity, determined by the test body and the rotation of the quantum vacuum on the basis of relation

$$\Omega = m\frac{(v+v_{qv})^2}{r}. \quad (57)$$

In [21] Pretorius and Khurana provided evidence that there are regions in the parameter space of the two body problem in general relativity where black holes evolve towards an unstable circular orbit, remain in that configuration for an amount of time sensitively related to the initial conditions, then either plunge towards coalescence or separate. For a one parameter family p of solutions interpolating between a deflection of a binary black hole, where for $p < p^*$ merger occurs promptly, while for $p > p^*$ it does not, they found that the unstable circular orbit is entered near the threshold of immediate merger at $p \approx p^*$ of parameter space. The number of orbits N observed near threshold is of the order

$$e^n \propto |p - p^*|^{\gamma}. \quad (58)$$

Moreover, in [22], Mondal, Rahaman and Singh have computed the equations of Lyapunov exponent and reciprocal of critical exponent

for both the cases of timelike and null-circular geodesics for rotating black hole and for special cases.

Now, in our approach of 3D DQV, by taking account of the Pretorius and Khurana calculation and the Mondal-Rahaman-Singh results, the parameter γ which characterizes how the number of orbits increases as the logarithmic distance to the critical parameter decreases, may be expressed as

$$\gamma = \frac{r^2}{2\pi}\left[3r^2\Delta + \frac{4mr^2}{(v+v_{qv})^2}\left({R_0}^2\frac{(v+v_{qv})^2}{r} - 4ma\frac{v+v_{qv}}{r} - r + 2m\right)\right]^{-1/2} \quad (59)$$

where m is the total mass of the black hole, $a = (v + v_{qv})r$, $\Delta = r^2 + a^2 - 2mr$, $R_0^2 = r^2 + a^2(1 + 2m/r)$. Equation (59) means that the Lyapunov exponent identifying the instability timescale associated with this geodesics is tightly linked with the velocity of the quantum vacuum and thus is directly determined by the fluctuations of the quantum vacuum energy density. The approach based on equation (59) may open interesting perspectives in the description of the instability of particular geodesics such as equatorial circular geodesics for time-like and null-circular geodesics, as well as in the definition of "boundary orbits" which lie between dynamically stable and unstable orbits (which start asymptotically closed to unstable circular orbits and end with spiralling in and out about the central black hole). In particular, if in Mondal's, Rahaman's and Singh's model, null circular geodesics have larger angular frequency than the time circular geodesics, on the basis of our approach, this result can be seen as a consequence of the bigger action of the velocity of the quantum vacuum, and thus of the fluctuations of the quantum vacuum energy density, in the null circular geodesics than the time circular geodesics. As regards these important topics, further research will give you more information.

5. Conclusions

The two-body problem and space-time frame-dragging effect can be elegantly described as a result of the dynamic quantum vacuum force acting on a given stellar object. This model has besides its mathematical model also physical explanation and in this sense can be considered as an improvement of the general relativity model.

References

1. A. Einstein, *Sitzungsberichte der Preussischen Akademie der Wissenschaften* (1915) 831.
2. C. Rovelli, *Relatività generale*, Adelphi, Milano (2021).
3. G.V. Kraniotis and S.B. Whitehouse, Classical and Quantum Gravity, 20 (2003), 4817-4835.
4. K.A. Hall, Exact results for the Kepler problem in general relativity. arXiv:0807.4109v2 [gr-qc] (2008).
5. D. Fiscaletti and A. Sorli, Space-time curvature of general relativity and energy density of a three-dimensional quantum vacuum. Annales UMCS Sectio AAA: Physica, LXIX (2014), 55-81.
6. D. Fiscaletti, *The timeless approach. Frontier perspectives in 21st century physics*, 1st ed., World Scientific, Singapore (2015).
7. D. Fiscaletti, About dark energy and dark matter in a three-dimensional quantum vacuum model. Foundations of Physics, 46, (2016), 1307-1340.
8. D. Fiscaletti and A. Sorli, About a three-dimensional quantum vacuum as the ultimate origin of gravity, electromagnetic field, dark energy ... and quantum behaviour. Ukrainian Journal of Physics, 61, (2016) 413-431.
9. D. Fiscaletti and A. Sorli, Dynamic quantum vacuum and relativity. Annales UMCS Sectio AAA: Physica, LXXI (2016), 11-52.
10. D. Fiscaletti and A. Sorli, Quantum vacuum energy density and unifying perspectives between gravity and quantum behaviour of matter. Annales de la Fondation Louis de Broglie, 42 (2017), 251-297.
11. D. Fiscaletti and A. Sorli, Quantum relativity: variable energy density of quantum vacuum as the origin of mass, gravity and the quantum behaviour. Ukrainian Journal of Physics, 63 (2018), 623-644.
12. K. Vankov, Particle orbits in general relativity: from planetary Solar system to black hole environment. https://hal.archives-ouvertes.fr/hal-01571766v2 (2018).
13. L.I. Schiff, Physical Review, 4 (1960), 215-217.
14. R.J. Adler, The three-fold theoretical basis of the Gravity Probe B gyro precession calculation. arXiv:1405.5511v1 [gr-qc] (2014).
15. A.S. Sorli, S. Celan, Schwarzschild energy density of superfluid quantum space and mechanism of AGNs' jets. Advanced Studies in Theoretical Physics, 15 (2021), 9-17.
16. N.J. Cornish, "Chaos and gravitational waves", Physical Review D, 64 (2001), 084011.

17. M. Mondal, P. Pradhan, F. Rahaman and I. Karar, Geodesic stability and quasi normal modes via Lyapunov exponent for Hayward black hole. Modern Physics Letters A, 35 (2020) 2050249.
18. J. Levin, Gravity waves, chaos and spinning compact binaries. Physical Review Letters, 84 (2000), 3515.
19. C.P. Dettmann, N.E. Frankel and N.J. Cornish, Fractal basis and chaotic trajectories in multi-black-hole spacetimes. Phys. Rev. D, 50 (1994), 618-621.
20. A.K. Yadav, M. Mondal and F. Rahaman, Singularity-free nonexotic compact star in f(R, T) gravity. Pramana Journal of Physics, 94 (2020), 90.
21. F. Pretorius and D. Khurana, Black hole mergers and unstable circular orbits. Classical and Quantum Gravity, 24 (2007), S83-S108.
22. M. Mondal, F. Rahaman and Ksh. Newton Singh, Lyapunov exponent, ISCO and Kolmogorov-Senai entropy for Kerr-Kiselev black hole. European Physical Journal C (2021), 81-84.

HADRONIC JOURNAL 45, 97 - 116 (2022)
DOI 10.29083/HJ.45.01.2022

NEW THERMODYNAMICS: THE SECOND LAW BURIED BY ILLUSIONS

Kent Mayhew
68 Pineglen Cres.
Ottawa, Ontario, Canada K2G 0G8

kent.mayhew@gmail.com

Received April 4, 2022
Revised April 7, 2022

Abstract

Reality involves inelastic collisions, thus one must consider both a system's radiation and its kinematics, as part of any system's energy. Since most closed gaseous system's energy is dominated by its kinematic energy, one can often ignore its radiation energy. A closed system's energy is recycled back into the system via its walls, hence allowing statistical thermodynamics to be used as a mathematical approximation, i.e., creates the illusion of elastic collisions. This illusion exists in most experimental systems, thus reenforcing traditional theory's considerations, all of which are based upon idealistic elastic collision. In open systems there is no complete recycling of energy. Hence, the statistical based approximations can lose its validity. It is in open systems where perpetual motion does not exist, which has been traditionally explained in terms of the second law, a law whose mathematical construct is limited to systems where the illusion of elastic collisions exists. Note that other explanations will be presented. Other, illusions do exist. For example, the illusion of reversibility prevents us from realizing that the magnitude of the latent heat of vaporization is greater than that of the latent heat of condensation. This also explains Loschmidt's paradox. A new understanding for Boltzmann constant will be given.

Keywords: Boltzmann's constant, Inelastic collisions, Kinematic heat transfer, Radiative heat transfer, Second Law.

Copyright © 2022 by Hadronic Press, Inc., Palm Harbor, FL 34684, USA

1. *Introduction*

The currently accepted mathematical description of the second law remains as follows [1 pg283]: "An equilibrium macrostate of a system can be characterized by a quantity S (called entropy), which has the following properties:

i) In any infinitesimal quasi-static process in which a system absorbs heat, its entropy changes by an amount:

$$dS = dQ/T \qquad 1$$

where T is a parameter characteristic of the macrostate of the system and is called its absolute temperature.

ii) In any process in which a thermally isolated system changes from one macrostate to another, its entropy tends to increase, i,e.,

$$\Delta S \geq 0 \text{ "} \qquad 2$$

This author has challenged any notion that an expanding system here on Earth can be deemed isolated [2]. In other words, a system's isobaric volume increase involves lost work (W_{lost}) [2-7]:

$$W_{lost} = (PdV)_{atm} \qquad 3$$

The subscript "atm" is all important [4], because it informs one that this lost work is done onto the expanding system's surrounding atmosphere. Accept that the atmosphere has mass. Then any expanding system on Earth's surface must lift that mass against gravity. This becomes a potential energy increase of its surrounding atmosphere, an energy that is forever lost by the expanding system, as it disperses throughout our atmosphere.

Without the clarity of the subscript "atm", one might consider PdV as being part of the expanding system. Thus, when considering an isothermal isobaric expanding system, one often writes:

$$TdS = dE + PdV \qquad 4$$

Clarity requires that one writes eq. 4 in terms of the expanding system (subscript "sys") [4] as follows:

$$(TdS)_{sys} = dE_{sys} + (PdV)_{atm} \quad 5$$

If the expanding system only performs lost work onto the surrounding atmosphere, then $dE_{sys} = 0$. If the system powers a device, then its work term must be added to the R.H.S. of eq. 5.

Quasi-static expansion often starts and finishes at atmospheric pressure hence, $P_{sys} \cong P_{atm}$. Furthermore, the expanding system is a subsystem within the atmosphere (as a system). Therefore, $dV_{sys} = dV_{atm}$. This does not validate writing:

$$TdS)_{sys} = dE_{sys} + (PdV)_{sys} \quad 6$$

To write eq. 6 is to incorrectly believe that work is internal to the expanding system instead of being work done onto its surroundings. It lends itself to incorrectly inserting, $PdV_{sys} = d(PV)_{sys} - VdP_{sys}$ into eq. 6. One can only insert $PdV_{atm} = d(PV)_{atm} - VdP_{atm}$ into eq. 5. However, this accomplishes nothing because the surrounding atmosphere remains isobaric hence, $dP_{atm} = 0$.

The above is why this author has challenged the conceptualization of free energies. See [4] for explanations concerning a new understanding of free energies.

Dealing with Clausius' notion of isothermal entropy change (*TdS*) is not as simple as challenging the accepted notions using lost work. For example, the difference in energy requirements between isometric and isobaric heat capacities can be explained in terms lost work, see [5].

Planck acknowledged that there are two ways to formulate thermodynamics [8]:

1) "We may take for granted the correctness of the mechanical view of nature, and assume that all changes in nature can be reduced to motions of material points between which there act forces which have a potential. Then the principle

of energy is simply the well-known mechanical theorem of kinetic theory, generalized to include all natural processes."

Or

2) "...leave open the question concerning the possibility of reducing all natural processes to those of motion, and start from the fact which has been tested by centuries of human experience and repeatedly verified "..."no way possible to have perpetual motion"[8 pg40].

Acceptance of 2) has crafted the second law of thermodynamics into the supreme postulate! Expanding systems here on Earth are not isolated, because they do work onto their surroundings. Therefore, the second law should not even apply.

Such understandings do not alter the second law's origins, i.e., Lord Kelvin [9], "It is impossible to transform an amount of heat completely into work in a cyclic process in the absence of other effects", i.e., Planck's #2. This enables one to also accept Planck's stated "correctness of the mechanical view of nature" (Planck's #1). Seemingly, entropy and its accomplice the second law has been rendered into an overcomplication.

Planck's stated application of mechanics involves elastic collisions at the atomic/molecular scale. Elastic allows one to express all of a gaseous system's energy in terms of the truncated Hamiltonian [7] with each point mass in the ensemble having its position and velocity defined. Elastic gaseous collisions remain the embodiment for both traditional kinetic theory and Maxwell's speed distributions. Hence, the evolution of statistical thermodynamics.

Statistical thermodynamics embraces Boltzmann's microscopic viewpoint concerning the mechanical behaviour of atoms/molecules based on the averaging of molecular properties. The truncated Hamiltonian is transformed into a probabilistic basis for entropy, becoming a measure of a system's disorder. Boltzmann envisioned the proportional correlation between the entropy (a macrostate) of an ideal gas and the number of distinct microstates using a logarithmic relation of microstates (InW). It was actually Planck who provided the proportionality constant (k), which led to Boltzmann's entropy formula ($S = kInW$). The implication being that systems trend to states involving the greatest number of possible configurations.

Entropy has been bestowed with an array of meanings. Most definitions revolve around entropy signifying a system's disorder, i.e.., "randomness of matter in incessant motion" [10]. Other definitions include entropy (S) is "the dispersal of a system's molecular energy" [11], or "S is a measure of the quality of that energy; low entropy implies higher quality, while high entropy implies lower quality" [12].

Entropy's ambiguity leaves the second law in a similar state. If intermolecular collisions result in the gas' dispersal, why call it an "entropy increase" instead of "dispersal?"

In an effort to delimit the Clausius entropy principle, in 1894 Planck wrote [13]: "It is completely unfounded, simply to assume that changes in Nature always proceed in the direction of lesser to greater probability." Seemingly, years later Planck somewhat altered his position.

The debate concerning to what degree probability governs the universe, and whether or not quantum mechanics is actually a complete theory continues on into the 21st century. If Planck was right in 1894, the currently accepted notions concerning the second law are not right. Does this also mean that the EPR (Einstein, Podolsky and Rosen) warrants more consideration, as recently done [14]?

Second law challenges are documented [15]. Challenges extend to recent considerations involving the Meissner Effect in superconductivity [16]. Daniel Sheehan [17] rightfully wrote that, "The second law of thermodynamics is an empirical law. It has no fully satisfactory theoretical proof. This being the case, its absolute validity depends upon its continued experimental verification in all thermodynamic regimes." The arguments presented herein renders any concept of the second law being empirically validated, as superficial, i.e., its verification is illusionary.

2.1 *Inelastic Collisions*

The truncated Hamiltonian and all that followed are limited to mechanical systems undergoing elastic collisions. It all collapses into scientific approximations, when one realizes that atomic/molecular collisions tend to be inelastic, which is the belief of this author [7]. Inelastic molecular (both inter and

intra) collisions means that photons/phonons must be created in order for energy to be conserved. An ensemble of such collisions will result in a spectrum of radiation, e.g., a blackbody spectrum [7,18-21].

This helps explain blackbody spectrums and can be extended to most bodies above absolute zero. It removes the requirement for blackbody radiation residing in enclosures surrounded by crystalline walls, e.g., a Jean's cube. A requirement based upon Planck's mathematics which adverted the UV catastrophe, but does not actually allow for the witnessed blackbody radiation in open systems.

One must therefore contemplate that a gaseous system contains both radiation and kinematic energies. In most gaseous systems the total energy associated with radiation generally will be minute in comparison to the total energy associated the gas's kinematic energies, e.g., vibrational, rotational and translational energies. Therefore, one can often approximate a gaseous system's total energy in terms of its kinematic energies. Interestingly traditional kinetic theory requires the molecular collisions to be elastic, while this author's kinetic theory allows for inelastic collisions. Note that this author's kinetic theory is a superior fit to known findings [22,23].

Maxwell's distributions serve as great approximations for what one often witnesses, e.g., closed system gaseous experiments at low to mid concentrations. Those being systems where the illusions of elastic collisions exist. This helps to explain why Maxwell's distributions fail when applied to indistinguishable particles at high densities [24 pg 11]. Specifically, high densities mean that interparticle collisions dominate over wall-particle collisions, hence the walls no longer impose their kinematics upon the particles, e.g., colliding gas molecules. Imposition being an assessment based on this author's kinetic theory [22,23].

The illusion of elastic collisions exists because the experimental system's walls recycle the thermal energy back into the closed system. This includes the system's radiative and kinematic energies. Specifically, the walls reflect, and/or absorb followed by radiate the energy back into the system. The walls also recycle the gas' kinematic energy by absorbing that energy and then imposing its kinematics back onto the gas molecules. This fulfills the illusions' requirements [7,18].

Accepted elastic collision mathematical structures, (e.g., in statistical thermodynamics), become valid approximations for describing sufficiently-dilute closed systems, e.g., experimental gaseous systems. However, to make universal assertions, such as the second law, based upon approximations becomes dubious at best. Importantly, a universal law must be equally applicable in describing open systems as it is in approximating closed systems and this is simply not the case. Open systems tend to lose energy as radiation vanishes into their horizons [7,18].

Before Boltzmann's statistical thermodynamics there was James Clerk Maxwell's Stosszahlansatz (molecular chaos hypothesis) [25]. Both are brilliant mathematical deductions based upon assumptions:

- The colliding molecules/atoms can be treated as point particles.
- The velocities of such colliding point particles are not correlated.
- Their velocities are independent of both their position and origins.
- The point particle collisions are elastic.

These assumptions allow the probability of a pair of colliding particles to be calculated by considering each particle separately, allowing a two-particle distribution function to be reduced to a product of one-particle distributions [25]. Hence, enabling one to ignore the probability of finding one particle with velocity $(\vec{v}_1)$ and another with $(\vec{v}_2)$ in an infinitesimal region. Thus, the particle's energy can be computed because they obey a new random distribution.

Such conjecture allows for a simplification of an overly complex math. The limitations that these assumptions impose should not be ignored out of convenience just so that a comprehensible math can be rendered. Reality holds that molecules are not spherical point particles, i.e., dimensionless. This alone implies that statistical thermodynamics (in its current state) is not the claimed all-inclusive theory.

From a thermodynamic perspective, real world inelastic collisions means that both Maxwell's molecular chaos hypothesis and Boltzmann's statistical thermodynamics suffer the fate of being approximations. They remain applicable approximations for systems where the illusion of elastic collisions exists, e.g., closed experimental systems.

Probability still plays a role in our universe but, the science needs a rethink. Interestingly, Dr Santilli has developed Hadronic math (iso-mathematics and geno-mathematics) [26-29] that could be beneficial in helping to describe irreversible thermodynamic processes such as those undergoing inelastic collisions. The reality may be that the complexities involved in all such processes, may simply be too much for complete mathematical lucidity.

2.2 *Absorption of Photon's Momentum*

Acceptance that blackbody radiation is a result of inelastic molecular collisions, then there must exist a mechanism by which molecules absorb radiation. This author has described the wave-particle duality in radiative heat transfer [18].

Specifically, a photon can act as:

- A wave, i.e., exchanging energy with molecules that possess lopsided charge distributions, as is witnessed in infrared spectrometry.
- A particle, i.e., pass its momentum onto atoms within molecules, thus adding to the molecule's vibrational energy in closed systems, i.e., thermal photons [18,30]. In open system's thermal photons with a preferred net direction can add to a gas molecule's translational, rotational and/or vibrational energies [30].

The above is not witnessed in infrared spectrometry because such information is lost when the blackbody spectrum is subtracted from their resultant spectrums. This subtraction leaves only the spectrum associated with photons acting as waves [18].

Heat capacities of gases as described in kinetic theory, can be mathematically defined in relation to Boltzmann's constant, e.g., this author's improved kinetic theory, which is a superior to fit known heat capacities [22,23], and allows for inelastic collisions.

In kinetic theory (both traditional and this author's) there is no frequency dependence, as infrared spectrometry-based theory implies. Realizing that atoms in molecules can absorb a thermal photon's momentum provides a universal

understanding for radiative heat transfer, that can be associated with Boltzmann's constant.

The thermodynamics of open systems can be different than that witnessed in closed systems, e.g., experimental systems [30]. What is of interest is the reality that experiments have shown that there is an undeniable absorption of our Sun's insolation at all wavelengths by the gases in our open system atmosphere [29].

2.3 *IRS, IBS and Hadronic Math*

The absorption and radiation of photons as particles/heat may help explain both the redshift and blueshift of witnessed spectrums in cosmology. Specifically, some witnessed spectrums (in both cosmology and our atmosphere) may be explained by the partial absorption of some local photon's momentum. This would be followed by the emission of radiation by gases through which the spectrum of photons passed through.

A plausible theoretical math for describing redshifts and blueshifts due to the absorption and ensuing radiation is the *Santilli IsoShift law* (developed by Dr Santilli) [29]. The law considers:

1) "*IsoRedShift* (IRS)"..."representing the loss of energy $E = \hbar w$ " by"electromagnetic waves to a medium at sufficiently low temperature". [29]
2) "*IsoBlueShift* (IBS)"..."representing the acquisition of energy $E = \hbar w$ by electromagnetic waves from a medium when at sufficiently high temperature." [29]
3) "*NoIsoShift* occurring for a media at intermediate temperature between IRS and IBS". [29]

Santilli's considerations were formulated prior to this author's assertion that photons can act as particles [18]. Any notion that energy is lost or acquired can be misleading, as energy is neither lost nor gained. Only the frequencies can change.

Specifically, parts of a spectrum of incident photons (blackbody?) could be absorbed by an ensemble of polyatomic gas molecules becoming part of the gas's

vibrational energy. This ensemble also emits a spectrum that is a function of the gas' temperature, e.g., a blackbody spectrum.

If the gas is comparatively cold, then that gas will emit a spectrum of photons at a lower mean frequency than it absorbs. Energy remains conserved because the colder gas emits a greater number of photons than it absorbs. Conversely, if the gas is comparatively hot, then that gas will emit a spectrum of photons at a higher mean frequency than it absorbs. Energy remains conserved because the hotter gas emits a lower number of photons than it absorbs [30].

Any ensemble of polyatomic gas molecules (above absolute zero) would absorb and emit photons that are relatable to the gas' temperature. This is irrelevant to whether or not a spectrum of incident photons from an external source is present. Proclaiming to completely know the true origins of the energy witnessed in some spectrums, may also be illusional. Note that the notion of temperature in open systems should also be questioned [30].

As previously stated, accepted statistical mechanics considers particles to be dimensionless entities undergoing elastic collisions. This serves well in approximating closed experimental systems where the illusion of elastic collisions exists. Santilli's Hadronic math should serve as a superior approximation for open system's where irreversibility is indisputable. Approximation being the definitive word.

The implications to the sciences extend far beyond thermodynamics, with ramifications throughout both atmospheric physics, e.g., what a spectrum really means, and cosmology, e.g., blueshifts that have led to debatable theorization concerning dark energy [29]. New understandings bring new questions.

2.4 *Illusion of Reversibility*

A reversible process is considered isentropic thus such processes can equally proceed forwards or backwards. Based on the current second law, processes where entropy increases are irreversible, i.e., processes that cannot readily return to their original state without an input of resources. Irreversibility is our common-sense reality for most real processes, while reversibility can be illusional.

Consider the heating of a liquid below a piston-cylinder, as shown in Fig.1. The expanding gas in the piston-cylinder performs lost work [$(PdV)_{atm}$]. The energy in (Q_{in}) can be defined by the following:

$$Q_{in} = dE_{sys1} + (PdV)_{atm} \qquad 7$$

Note that dE_{sys1} includes the following:

- Energy required to increase the temperature of the liquid [$(nC_v dT)_{liquid}$]
- Energy required to increase the temperature of the gas [$(nC_v dT)_{gas}$]
- Latent heat of vaporization [$dU_{sys1} + (PdV)_{atm}$], where dU is the change to the bonding energy.

Although overly idealistic, imagine that both the heating and expansion of System 1 was quasi-static. Theoretically, such a process could be deemed isothermal (dT=0), hence eq. 7 becomes:

$$Q_{in} = dU_{sys1} + (PdV)_{atm} \qquad 8$$

Since $(PdV)_{atm}$ is lost work, then the heating was an irreversible process. Specifically, if one were to turn off the heat (Q_{in}= 0), heat/energy would slowly radiate through the piston and dissipate into the surroundings. Thus, the system would contract.

From a mechanical perspective the piston has returned to its preheated state. Hence, the casual observer may conclude that this was a reversible process. However, if one magically collected all the dissipating heat that radiates through the piston cylinder, then only dU_{sys1} could be collected. The illusion of reversibility in boiling is based upon mechanical equilibrium, rather than any precise measurement of energy, specifically that associated with condensation.

2.5 *Latent Heat and Loschmidt's Paradox*

Eq. 8 can be related to the latent heat of vaporization, i.e., [4]:

$$L_{(l\rightarrow g)} = dU + (PdV)_{atm} \qquad 9$$

where "L' signifies the latent heat, "$(l \rightarrow g)$" indicates that the change is from liquid to gas, (i.e., vaporization), and "dU" is the change in bonding energy.

The latent heat of vaporization is readily measured in an isobaric calorimeter. Conversely, the latent heat of condensation cannot be easily measured. This, plus the illusion of reversibility helps explain why the following has not been realized.

Lost work [$(PdV)_{atm}$] remains irreversible, thus the latent heat of condensation must be [4]:

$$L_{(g \rightarrow l)} = -dU \qquad 10$$

where "$(g \rightarrow l)$" signify condensation.

Since, $L_{(l \rightarrow g)} > L_{(g \rightarrow l)}$, [7] the latent heat suffers from the illusion of reversibility. This also forms the basis of Loschmidt's paradox (A.K.A., the irreversibility paradox). Johan Josef Loschmidt's paradox (1876) placed the time reversal of fundamental macroscopic system's processes at odds with the second law. The claim being that isolated systems tend to arbitrarily return to their initial conditions hence the second law appears paradoxical.

However, the system was never fully isolated. Not only was lost work bestowed into the surrounding atmosphere, but heat was also able to radiate back out through the piston. Such simple understandings remove the need for both the second law and any paradoxes. Furthermore, it removed the need of Loschmidt's challenges to Boltzmann's H-theorem, which was based upon the entropy increase of non-equilibrium gases.

It should be stated that there are inefficiencies, and that equations 9 and 10 assume 100% efficiency (see Section titled "Engines…").

2.6 *Emphasis Concerning Lost Work*

For emphasis: When any higher-pressure System A performs work on any lower-pressure System B, then either, and/or:

- System B experiences a temperature increase.
- System B experiences an isometric pressure increase.
- System B experiences a potential energy increase.

Lost work being potential energy increase of the surrounding atmosphere, cannot be recovered. When the heat is shut off in Fig. 1, the atmosphere's mechanical weight drives the piston back to its original preheated position.

If the atmosphere performed work onto System 1, as System 1 contracts, then either its pressure, temperature or potential energy would increase. Rather, the piston simply returned to its preheated position, i.e., System 1 returned to its preheated state.

One could hide the above reality with a charade of complicity, such as arguments based upon degrees of freedom, or even entropy. Such complications do not enlighten the issue.

2.7 *Thermal Reversibility*

Consider the system shown in Fig. 2. A fire heats System 1 (Q_1), which in turn increases the thermal energy within System 2 (Q_2). In this case, heat (Q_{lost}) dissipates through the walls of System 2 into the surrounding atmosphere. For such a process:

$$Q_1 = Q_2 + Q_{lost} \qquad 11$$

In Fig. 2 there is an obvious direction for heat flow. Certainly, when Q_{lost} enters the surrounding atmosphere, it disperses throughout the atmosphere increasing the atmosphere's temperature by an infinitesimal amount, i.e., a heat sink. Since, $T_{sys2} \geq T_{atm}$, Q_{lost} cannot be magically collected and returned back into System 2.

The only time that the atmosphere can transfer heat back into a system is when the atmosphere's temperature is greater than that system's temperature.

Therefore, the process shown in Fig. 2 and described by eq. 11 is an irreversible process. For the process to be perceived as reversible then, Q_{lost}=0. Note that no work was done.

As rudimentary as this is, it demonstrates that reversibility cannot involve any real dispersal of thermal energy into some cooler massive surroundings, e.g., heat sink, where concentrating thermal energy back into its original state requires an input of resources.

2.8 *Engines and Perpetual Motion*

One should now realize why no one has actually managed to build Sadi Carnot's idealistic engine, which assumes that its cycle is fully reversible. Obviously, even if an engine was perfectly designed, it would experience lost work during any step involving expansion. It also would suffer the fate of energy dissipation/dispersal during any process of heat transfer. Therefore, the want to explain such things in terms of the second law becomes an overcomplication.

Engines often function as alternating closed and open systems. As a closed system one could approximate a piston-cylinder's contents in terms of statistical thermodynamics. That being the act of thinking in terms of elastic collisions and contemplating the engine's energy only in terms of its gaseous content's kinematics.

One has to be careful when making mathematical assertions and then claiming closed system experiments as verification. Traditional descriptions should not be universally applied, especially when considering open systems, such as those where we tend to reside.

Friction, both internal and external, e.g., drag, is experienced by all moving objects. This, combined with lost work by expanding systems, becomes an inarguable factor in explaining why perpetual motion cannot occur, at least here on Earth. In both cases, the movement of matter within our atmosphere lends itself to the infinitesimal heating of the atmosphere. Note that the impossibility of perpetuum mobile was first noted by Carnot in 1824 [16].

Inefficiencies also arise due to the reality that not all of a gas' kinematic energy can be extracted from a gaseous system and then be used to perform work, i.e., only 2/3 of a gas's kinematic energy can be used to perform work [7].

Furthermore, heat tends to dissipate hence its exchange is rarely reversible. This applies more to radiation than kinematic energy but in open systems with no constraints, e.g., gravity, it applies to both.

Perpetual motion remains idealistic rather than realistic. Issues with perpetual motion can be better understood without the second law, at least in the way that it is currently described.

2.9 *Boltzmann's Constant*

Boltzmann's entropy formula remains a foundation for mathematical approximations with Boltzmann's constant (k) being its proportionality constant. Is there another simpler explanation?

An understanding can be found by rewriting the ideal gas law ($k=PV/NT$) and then differentiating. Followed by treating processes as isobaric. One then obtains:

$$k = (PdV)_{atm}/NdT \qquad 12$$

Boltzmann's constant becomes the proportionality constant for the amount of lost work an expanding gas does, per molecule, per unit temperature change.

One with vested interest in traditional dogma should at least realize that eq. 12 is structured around isobaric volume increases, as are accepted conceptualization for entropy increases. In other words, the association of the increase in possible number of energy states with a system's volume increase, gains clarity. Note that the traditional assertion of limiting changes to the possible number of states, to a system's volume increase, troubles this author.

The above insights challenge entropy's association with randomness as some scientific foundation. Since randomness is an obscure concept, its association is not overly scientific [31].

The issues discussed herein, can be extended to the mathematical constructs upon which statistical thermodynamics is based. This brings into question the

very notions behind the most ambiguous of all parameters, that being entropy itself.

Conclusions

Does one remain steadfast in the notion that all systems can be explained using accepted statistical thermodynamics, thus, continuing to believe that this probability-founded description is the only one that counts? It is inarguable that closed system experimentation seemingly verified this approach, which reenforced the path to the current second law. Certainly, the second law has been challenged. Each challenge unwittingly discredits the foundations of statistical mechanics.

One can devise complex arguments either for, or against, the second law. However, simple arguments remain the best. Accepting that molecular collisions are inelastic, opens a pandora's box. It challenges the fundamentals upon which the second law was built. That being statistical thermodynamics as a universally applicable mathematical structure all based upon elastic collisions.

Inelastic molecular (both inter and intra) collisions explain blackbody radiation in ways that it never has been. This includes that which surrounds gas molecules, hence constitutes part of a gaseous system's energy. This is not part of the energy that statistical mechanics describes. The system's radiation's energy is generally minute in comparison to its kinematic energy. Thus, statistical mathematics remain a valid approximation when describing the energy contained within closed gaseous systems, e.g., most experimental systems.

Accepting that blackbody radiation can be associated with inelastic molecular collisions, then there must be a mechanism by which molecules absorb thermal radiation. This can be explained by thermal photons acting as particles, i.e., atoms in molecules absorbing a photon's momentum.

A closed system's walls recycle its radiation energy by reflection or by the absorption followed by its emission back into the system. The system's walls also impose their kinematics onto the enclosed gas molecules. This combination forms what this author has described as the illusion of elastic collisions, which has emboldened the approximation of a system's energy using statistical thermodynamics. A result being the second law.

Other illusions exist. The reality is that expanding systems must lift the weight of their surrounding atmosphere. This becomes lost work [$W_{lost}=(PdV)_{atm}$] by an expanding system, resulting in an atmospheric potential energy increase. A potential increase that is lost when it disperses throughout the atmosphere. A potential energy increase that is not readily noticed by one observing an expanding system.

Importantly, when the heat that caused the expansion is turned off, then the atmosphere can mechanically drive the system back to its original state. Thus, giving an illusion of reversibility that also being the basis for Loschmidt's paradox.

Our inability to measure dissipating/dispersing energy has prevented the realization that here on Earth's surface expanding systems are irreversible. If one could measure such energy, then one should realize that expanding systems undergo lost work. In which case one should:

- Abscond the currently accepted second law.
- Realize that the magnitude of the latent heat of vaporization is greater than the magnitude of the latent heat of condensation. The difference being $(PdV)_{atm}$ when assuming 100% efficiency.

Furthermore, we presented a new simpler perspective for Boltzmann's constant (k) that being a proportionality constant for lost work per molecule per unit temperature increase.

Mathematics can be a wonderful descriptor for both reality and theories. Mathematics, as the foundation for theorization can become disingenuous. Arguably, such is the foundation for both entropy and its accomplice the second law of thermodynamics, as other explanations exist

Acceptance that a science is an overcomplication may not suit those with vested interests. Nikulov rightfully points out that: "Most scientists have to understand that the supreme position of the second law has no scientific justification" [16]. Perhaps realizing that the second law, along with its mathematical foundations have been established upon illusions, may alleviate any heartfelt anguish.

Figures

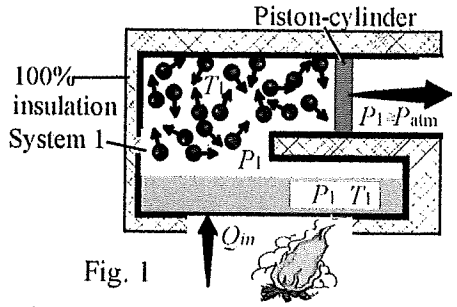


Fig. 1

Fig. 1. Heating the liquid causes a mass transfer from the liquid to the vapor state resulting in the piston-cylinder's isothermal expansion.

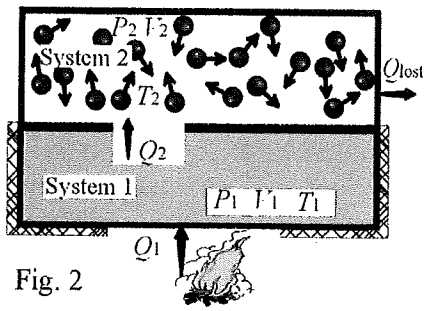


Fig. 2

Fig. 2. Shows the heating of System 1 (Q_1), with System 2 extracting heat from System 1 (Q_2). Energy is lost by System 2 (Q_{lost}) as it dissipates through its walls into its surroundings.

Acknowledgements

I would like to thank Jack Denur (associate of North Texas University), plus Lloyd and Gail Mayhew for their assistance in putting this paper together.

References

[1] Reif, F., "Fundamentals of Statistical and Thermal Physics", McGraw-Hill, New York, 1965
[2] Mayhew, K.W., "Second law and lost work", Phys. Essays **28**, vol 1,152 (2015)
[3] Mayhew, K.W., "New Thermodynamics: Inefficiency of a Piston-Cylinder", EJERS Vol 5, 2, (2020) pg 187-191

[4] Mayhew, K.W., "New Thermodynamics: Reversibility and Free Energy", Hadronic Journal, Vol 43, 1, 2020 pg. 51-60
[5] Mayhew, K.W., "New Thermodynamics: Reversibility, Entropy and Adiabatic Processes" E-J Physics, Vol 2 Issue 2, (2020)
[6] Mayhew, K.W., "Entropy an ill-conceived mathematical contrivance" Phys. Essays **28** vol 3, 352 (2015)
[7] Mayhew, K.W. "New Thermodynamics: Inelastic Collisions, Lost Work and Gaseous Inefficiency". Hadronic Journal vol 44 (1) March 2021, pg. 67-97
[8] Planck, Max, "Treatise on Thermodynamics", Longman, Green and co, London, 1917
[9] Dunning-Davis, Jeremey "A Cautionary Note on the Second Law of Thermodynamics" Presented in 2010 a
[10] Tilley D.E., "Contemporary College Physics" Benjamin Cummings Publishing Don Mills, Ont. Canada 1979
[11] Lambert F.L., "Entropy is simple, qualitatively". J Chem. Educ 79 187 (2002)
[12] Atkins , P,. "Four laws that drive the universe" Oxford University Press Oxford England 2007
[13] Hartmann C., "On the 150th birthday of Max Planck: On Honesty towards nature" https://21sci-tech.com/translations/Planck% 20150th%20birthday-Nov.%2014.pdf
[14] Santilli, R., M.., "Overview of the Einstein-Podolsky-Rosen Argument with Applications in Physics, Chemistry and Biology" APAV (2021)
[15] Capek, V., Sheehan D. P., "Challenges to the second law of thermodynamics", Springer Press, Netherlands 2005
[16] Nikulov, A., "The Law of Entropy increase and the Meissner Effect", Entropy, 24(1), 83 (2022)
[17] D. Sheehan, "Four Paradoxes Involving the Second Law of Thermodynamics" J. Sci. Exploration, 12 (2), 303 (1998)
[18] Mayhew, K.W., "New Thermodynamics: Wave-Particle Duality in Radiative Heat Transfer" Hadronic Journal vol 44 215-226 (2021)
[19] Mayhew, K.W., "New Thermodynamics: Inelastic Collisions, Blackbody Radiation, Entropy and Light", Vol 2, 6, pg 1-6
[20] Mayhew K.W., "New Thermodynamics: Temperature, Sun's Insolation, Thermal, and Blackbody Radiation", EJERS, Vol. 5, 3(2020) pg 264-270
[21] Mayhew, K.W., "New Thermodynamics: Inelastic Collisions and Cosmology", E-J Phys. Vol 2, 6, (2020) pg 1-7
[22] Mayhew, K.W., "A New Perspective for Kinetic Theory and Heat Capacity", Prog. in Phys., Vol. 13, 4 (2017) pg 166-173
[23] Mayhew, K.W., "Kinetic Theory: Flatlining of Polyatomic Gases", Prog. in Phys., Vol. 14, 2 (2018) pg 75-79
[24] Haddad, W. M., "A Dynamical Systems Theory of Thermodynamics", Princeton University Press, New Jersey, 2019
[25] https://en.wikipedia.org/wiki/Molecular_chaos Oct., 2021
[26] Dunning-Davies, J., "The thermodynamics Associated with Santilli's Hadronic Mechanics" Prog. in Phys., Vol. 4, (2006) pg 24-26
[27] Bhalekar, A.A., "Santilli's lie-admissible mechanics. The only option commensurate with irreversibility and nonequilibrium thermodynamics AIP Conference Proceedings 1558, 702, (2013)

[28] Santilli, R.M., "Lie-admissible invariant treatment of irreversibility for matter and antimatter at the classical and operator levels" " Il Nuovo Cimento V 121 B, N. 5 (2006) pg 443-486

[29] Ahmar, H., Amato,G., Kadeisvili, J.V., Manuel, J., West, G., Zorgorodniz, O. " Additional experimental confirmations of Santilli's Isoredshift and the consequential expected absence of the universe expansion" J. Comput. Methods Sci. and Eng. 13 (2013) pg. 321-357

[30] Mayhew, K.W., "New Thermodynamics: Questioning Temperature in Open systems, e.g., In Atmospheric Physics" ". Hadronic Journal vol 44 (1), (2021) pg. 431-444

[31] Ben-Naim, A., "A Farewell to Entropy: Statistical Thermodynamics Based on Information". World Scientific Publishing Co., London, New York